BUCKNELL UNIVERSITY LIBRARY
AUG 29 1955
LEWISBURG, PA.

ROYAL AERONAUTICAL SOCIETY 1866

Journal

of

THE ROYAL AERONAUTICAL SOCIETY

AUGUST 1955

CONTENTS

NOTICES

DEFINITIONS OF THE THRUST OF A JET ENGINE AND OF THE INTERNAL DRAG OF A DUCTED BODY

THE STRESS ANALYSIS OF THE CIRCULAR CONICAL FUSELAGE WITH FLEXIBLE FRAMES — W. J. GOODEY

SOME PROBLEMS ASSOCIATED WITH STRESS CONCENTRATION — H. L. COX

TECHNICAL NOTES

The Effect of Curvature on the Centre of Shear—W. JOHNSON. Buckling of Rectangular Plates Under Bi-Axial Compression—J. S. PRZEMIENIECKI. A Chart for Steady Circling Flight—E. SPONDER.

THE LIBRARY — REVIEWS — REPORTS

TEN SHILLINGS MONTHLY

LONDON

4 HAMILTON PLACE W1

RELIABILITY

HYDRAULIC COMPONENTS

Airframe hours flown	1450
Aircraft in service	12
No. of hydraulic components	1000
Components confirmed faulty	2

LANDING GEAR

Total number of landings to date	22,572

An oleo has never failed on service causing a collapse.

Good-very good but not good enough. HR.

int those concerned with the
.L.W. system in the [redacted]
nd landings of the [redacted] fleet
sted the hydraulic system under
ic components are overhauled
lent to 2,400 flying hours, but
components are the original
were new.

Our objective is ABSOLUTE RELIABILITY pursued by - Continuous research - Careful design-Vigorous testing-and precise manufacture

★ The facts quoted are taken from an internal Service report of one of our customers

ELECTRO-HYDRAULICS LIMITED
WARRINGTON

Journal of

THE ROYAL AERONAUTICAL SOCIETY

WITH WHICH IS INCORPORATED THE INSTITUTION OF AERONAUTICAL ENGINEERS

PUBLISHED MONTHLY AT

4 HAMILTON PLACE, LONDON, W.1

Telephone: Grosvenor 3515-9 *Telegrams*: Didaskalos, Audley, London

SUBSCRIPTIONS: £6 6S. 0D. PER ANNUM, POST FREE. SINGLE COPIES: 10S. 6D., POST FREE.

VOLUME 59 AUGUST 1955 NUMBER 536

CONTENTS

PAGE

Notices XLI

News of Members—Branch Area Meeting: Birmingham Branch—Associate Fellowship Examination—Graduates' and Students' Section: Visits—High Speed Photography Congress—Massachusetts Institute of Technology Foreign Student Summer Project 1956—Royal Aeronautical Society Staff Requirement—The Library—Diary—Journal Binding—Changes of Address—Elections—Reprinting Annual Reports and Journals.

Definitions of the Thrust of a Jet Engine and of the Internal Drag of a Ducted Body 517

The Stress Analysis of the Circular Conical Fuselage with Flexible Frames W. J. GOODEY 527

Some Problems Associated with Stress Concentration H. L. COX 551

Technical Notes 562

The Effect of Curvature on the Centre of Shear—W. JOHNSON. Buckling of Rectangular Plates Under Bi-Axial Compression—J. S. PRZEMIENIECKI. A Chart for Steady Circling Flight—E. SPONDER.

The Library 573

REVIEWS:—Gas Turbines for Aircraft—Fluid Dynamics of Jets—Airplanes of the World—Strömung Durch Rohre; Umströmung von Kropen bei Zweidimensionaler Strömung; Umströmung von Korpen bei Raumlicher Strömung—Jahrbuch 1953 der Wissenschaftlichen Gesellschaft fur Luftfahrt—The Observer's Book of Weather—The Continuum and Other Types of Serial Order; Almost Periodic Functions; Treatise on Thermodynamics; Advanced Dynamics of a System of Rigid Bodies; The Dynamical Theory of Gases—Manufacture and Application of Lubricating Greases—Lectures on Partial Differential Equations—Analysis of Deformation—The First and the Last—Spontaneous Ignition of Liquid Fluids—High Speed Aerodynamics and Jet Propulsion—Frontier to Space—Who's Who in World Aviation—The Airport Visitor and Air Traveller—Flexure Devices: Pivots, Movements, Suspensions—Continental Military Aircraft; and British Military Aircraft.

ADDITIONS TO THE LIBRARY. REPORTS.

All communications for publication in, and advertisements for, the JOURNAL *should be addressed to the Editor,* JOAN BRADBROOKE, *A.R.Ae.S.; all communications on general matters affecting the Society should be addressed to the Secretary,* A. M. BALLANTYNE, *T.D., B.Sc., Ph.D., Hon.F.C.A.I., A.F.I.A.S., A.F.R.Ae.S.*

None of the papers or paragraphs must be taken as expressing the opinion of the Council of the Royal Aeronautical Society unless otherwise stated.

PRINTED BY THE LEWES PRESS (WIGHTMAN & CO. LTD.), LEWES, SUSSEX, ENGLAND, AND PUBLISHED BY THE ROYAL AERONAUTICAL SOCIETY, 4 HAMILTON PLACE, LONDON, W.1, ENGLAND

THE ROYAL AERONAUTICAL SOCIETY

Overseas Divisions

AUSTRALIA

President: Professor A. V. STEPHENS, F.R.Ae.S.
Hon. Secretary: W. ISBISTER, [A.F.R.Ae.S.
19 Wallace Street, Burwood, Sydney, New South Wales.

NEW ZEALAND

President: C. W. LABETTE, A.F.R.Ae.S.
Vice-President: A/Cdre. G. CARTER, A.F.R.Ae.S.
Hon. Secretary: T. T. N. COLERIDGE, [A.F.R.Ae.S.,
The Shell Company of New Zealand Ltd., Box 2091, Wellington.

SOUTHERN AFRICA

President: Dr. R. W. S. CHEETHAM, A.F.R.Ae.S.
Hon. Secretary and Hon. Treasurer: A/Cdre. J. H. C. WAKE,
53/54 Norwich Union Buildings, 91 Commissioner Street, Johannesburg.

Overseas Branch

SINGAPORE *President*: Air Marshal Sir FRANCIS J. FRESSANGES, C.B.
Chairman: Mr. CRABBE.
Hon. Secretary: A. M. LOVATT, A.R.Ae.S., c/o Malayan Airways Ltd., Kallang Airport, Singapore.

Branches in the United Kingdom

BELFAST
President: Rear-Admiral Sir MATTHEW S. SLATTERY, C.B., [F.R.Ae.S.
Chairman: D. KEITH-LUCAS, F.R.Ae.S.
Hon. Secretary: A. R. DAFFEY, Grad.R.Ae.S.,
Short Brothers & Harland Ltd., Queens Island, Belfast.

BIRMINGHAM
President: O. N. LAWRENCE, A.F.R.Ae.S.
Chairman: R. ELLIS, A.R.Ae.S.
Hon. Secretary: C. P. HOMES, A.R.Ae.S.,
81 Peplins Way, Kings Norton, Birmingham, 30.

BOSCOMBE DOWN
President: Air Commodore R. A. RAMSEY RAE, O.B.E.
Chairman: J. HANSON, F.R.Ae.S.
Hon. Secretary: D. J. HIGTON, A.F.R.Ae.S.,
Ministry of Supply, A. & A.E.E., Boscombe Down, Amesbury, Wilts.

BRISTOL
President: Dr. S. G. HOOKER, O.B.E., F.R.Ae.S.
Chairman: R. HAFNER, F.R.Ae.S.
Hon. Secretary: R. E. MILLS,
Aircraft Division, No. 1 D.O., Bristol Aeroplane Co. Ltd., Filton, Bristol.

BROUGH
President: R. BLACKBURN, O.B.E., Hon.F.R.Ae.S.
Chairman: M. G. K. BYRNE, A.F.R.Ae.S.
Hon. Secretary: R. W. HILKEN, D.F.C., A.F.R.Ae.S.
Blackburn & General Aircraft Ltd., Brough.

CHESTER
President: J. C. CORBY, O.B.E.
Chairman: W. T. NEILL, M.B.E., A.R.Ae.S.
Hon. Secretary: R. Q. E. EDEN, A.F.R.Ae.S.,
Thornton Research Centre, P.O. Box 1, Chester.

COVENTRY
President: H. M. WOODHAMS, C.B.E., F.R.Ae.S.
Chairman: W. J. PETERS.
Hon. Secretary: C. T. SCULTHORPE, A.F.R.Ae.S.,
Sir W. G. Armstrong Whitworth Aircraft Ltd., Baginton, Coventry.

DERBY
President: LORD HIVES, C.H., M.B.E., Hon.F.R.Ae.S.
Chairman: A. G. ELLIOTT, C.B.E., F.R.Ae.S.
Hon. Secretary: W. R. BENDALL, A.F.R.Ae.S.,
Rolls-Royce Ltd., Derby.

GLASGOW
President: Professor W. J. DUNCAN, C.B.E., F.R.Ae.S.
Chairman: D. B. SWINBANKS, A.F.R.Ae.S.
Hon. Secretary: A. W. BABISTER, A.F.R.Ae.S.,
Dept. of Aeronautics, The University, Glasgow, W.2.

GLOUCESTER AND CHELTENHAM
President: LT.-GEN. SIR JOHN EVETTS, C.B., C.B.E., M.C.
Chairman: J. F. CUSS.
Hon. Secretary: R. C. DELL, A.R.Ae.S.
Dowty Equipment Ltd., Arle Court, Cheltenham.

HALTON
President: Marshal of the R.A.F. VISCOUNT TRENCHARD, Hon.F.R.Ae.S.
Chairman: Group Capt. E. E. KNOWLES, M.B.E.
Hon. Secretary: Flt. Lt. R. W. DURRANT, A.R.Ae.S.,
Apprentice Schools, R.A.F., Halton.

HATFIELD
President: Sir GEOFFREY DE HAVILLAND, C.B.E., A.F.C., Hon.F.R.Ae.S.
Chairman: C. A. PIKE, O.B.E., A.F.C., A.F.R.Ae.S.
Hon. Secretary: E. J. MANN, A.R.Ae.S.,
de Havilland Aircraft Co. Ltd., Hatfield.

HENLOW
President: Air Commodore C. L. DANN, C.B.E., F.R.Ae.S.
Chairman: Group Captain T. G. G. HUTCHINSON, C.B.E.
Hon. Secretary: Flight Lieut. A. W. CULMER, A.F.R.Ae.S.,
Engineer Wing, R.A.F. Technical College, Henlow, Bedfordshire.

ISLE OF WIGHT
President: Sir ARTHUR GOUGE, Hon.F.R.Ae.S.
Chairman: H. KNOWLER, F.R.Ae.S.
Hon. Secretary: L. W. ROSENTHAL, A.F.R.Ae.S.,
Saunders-Roe Ltd., Osborne, E. Cowes, Isle of Wight.

LEICESTER
Chairman: F. WATKIN, A.F.R.Ae.S.
Hon. Secretary: K. B. AYERS, A.F.R.Ae.S.,
10 Spinney Rise, Birstall, Leicester.

LUTON
President: Air Vice-Marshal Sir CONRAD COLLIER, K.C.B., [C.B.E.
Chairman: E. N. B. BENTLEY, A.F.R.Ae.S.
Hon. Secretary: H. J. BROOKS, A.F.R.Ae.S.,
Hunting Percival Aircraft Ltd., Luton Airport.

MANCHESTER
President: Sir ROY H. DOBSON, C.B.E., J.P., F.R.Ae.S.
Chairman: C. E. FIELDING, O.B.E., F.R.Ae.S.
Hon. Secretary: J. A. E. WATERFALL,
56 Manor Avenue, Ashton-on-Mersey, Cheshire.

MERTHYR TYDFIL
President: C. S. GARDNER.
Chairman: J. INGRAM.
Hon. Secretary: G. M. MAGILL, A.F.R.Ae.S.
Teddington Controls Ltd., Cefn Coed, nr. Merthyr Tydfil, South Wales.

PORTSMOUTH
President: A. TOWNSLEY.
Chairman: G. M. TWYMAN.
Hon. Secretary: E. M. BELLAMY, A.R.Ae.S.,
de Havilland Aircraft Co. Ltd., Airspeed Division, The Airport, Portsmouth.

PRESTON
President: Sir GEORGE H. NELSON.
Chairman: D. L. ELLIS, F.R.Ae.S.
Hon. Secretary: E. LOVELESS, A.F.R.Ae.S.,
Aircraft Division, English Electric Co. Ltd., Warton Aerodrome, nr. Preston.

READING
President: Sir FREDERICK HANDLEY PAGE, C.B.E., Hon.F.R.Ae.S
Chairman: R. J. FENNER, A.F.R.Ae.S.
Hon. Secretary: G. H. DAVIES, Grad.R.Ae.S.,
Handley Page (Reading) Ltd., The Aerodrome, Woodley, Reading.

SOUTHAMPTON
Chairman: H. C. SMITH, F.R.Ae.S.
Hon. Secretary: G. T. HUDSON, A.F.R.Ae.S.,
Folland Aircraft Ltd., Hamble, Hants.

WEYBRIDGE
President: G. R. EDWARDS, C.B.E., F.R.Ae.S.
Chairman: H. H. GARDNER, F.R.Ae.S.
Hon. Secretary: J. H. SINCLAIR, A.F.R.Ae.S.,
Vickers-Armstrongs Ltd., Weybridge.

YEOVIL
President: E. C. WHEELDON.
Chairman: D. L. HOLLIS-WILLIAMS, F.R.Ae.S.
Hon. Secretary: L. A. LANSDOWN, A.F.R.Ae.S.,
Westland Aircraft Ltd., Yeovil.

NOTICES

An annual sum of £250 is available for premium awards for papers, including Technical Notes, published in the Journal. Members and non-members of the Society are invited to submit papers on any aspect of aeronautics

News of Members

Dr. J. H. Argyris (*Fellow*), formerly Reader in the Theory of Aeronautical Structures, has been appointed to the newly created Chair of Aeronautical Structures at the University of London, tenable at Imperial College.

Group Captain A. F. Bandidt (*Associate*) has been appointed the Handley Page Representative in Australia, New Zealand and the Far East.

B. D. Blackwell (*Associate Fellow*) has been appointed Engineer-in-Charge of a new department for the development of turbine blades of the Engine Division of the Bristol Aeroplane Co.

P. Brett (*Associate Fellow*), formerly Chief Aerodynamicist of de Havilland Propellers Ltd., has been appointed Chief Project Engineer of the Company.

R. E. Catterwell (*Graduate*) has been awarded the James Clayton Fellowship by the Institution of Mechanical Engineers. The Fellowship is tenable at the College of Aeronautics, Cranfield.

J. R. Ewans (*Associate Fellow*), formerly Deputy Chief Designer, has been appointed Chief Designer to A. V. Roe & Co. in succession to Mr. S. D. Davies who recently resigned from that position.

B. S. Massey (*Associate Fellow*) has been appointed Assistant Chief Designer of the Engine Division of the Bristol Aeroplane Co.

Squadron Leader F. Murphy (*Associate*), formerly the Chief Production Pilot of Hawker Aircraft Ltd., has been appointed R.A.F. Liaison Officer (Flying) to Hawkers.

J. W. Piggott (*Associate Fellow*) has been appointed Apprentice Supervisor at the Dunlop Rim & Wheel Co. Ltd.

R. A. Read (*Associate*), formerly of Cardiff Airport, has been appointed Aerodrome Commandant at Bahrain, Persian Gulf.

Dr. A. E. Russell (*Fellow*) has relinguished the title of Chief Designer of the Bristol Aeroplane Co. and now holds the post of Chief Engineer (Aircraft Division).

B. Saravanos (*Associate Fellow*), previously with Handley Page (Reading), has taken up a post as Lecturer in Mechanical Engineering at Coventry Technical College.

E. R. Sisson (*Associate*) has been appointed Technical Service Manager to Smiths Aircraft Instruments Ltd.

S. S. Tresilian (*Associate Fellow*) has been appointed Assistant Chief Engineer of the Engine Division of the Bristol Aeroplane Co.

G. W. Underwood (*Associate*), formerly Structural Test Engineer at Gloster Aircraft Co. Ltd., has been appointed Deputy Chief of the Company's Structural Research Department.

Jeffrey Watkins (*Associate Fellow*) has joined Solar Aircraft Co. as Senior Aerodynamicist. He was formerly with Westinghouse Electric Corporation, Philadelphia.

Branch Area Meeting—Birmingham Branch

An Area Meeting is being held at the Birmingham Branch on 23rd September 1955. Mr. G. C. Best, Head Performance Engineer, Orenda Engines Ltd., Canada, will lecture on the "Application of a Digital Computor to Gas Turbine Engineering." The meeting will be held at the Chamber of Commerce, New Street, Birmingham, at 7.15 p.m.

Associate Fellowship Examination

Entry forms for Part II of the Old Syllabus of the Associate Fellowship Examination may be sent in up to 31st August 1955. Entries for the Revised Syllabus closed on 30th June.

Graduates' and Students' Section—Visits

A visit has been arranged to the Royal Air Force, West Malling, for Saturday 24th September. This is an Operational Fighter Station, within easy reach of London, and applications for the visit should be made immediately to the Hon. Visits Secretary, Mr. P. D. Stewart, B.O.A.C. Operations Development Unit, London Airport, Hounslow, Middlesex.

A visit has also been arranged to the National Aeronautical Establishment at Bedford for Wednesday 12th October. It is hoped to arrange for a coach party to travel from London for this visit, and in order to complete these arrangements, early application should be made to the Hon. Visits Secretary as above.

High Speed Photography Congress

The Third International Congress on High Speed Photography will be sponsored by the Department of Scientific and Industrial Research and will be held in London from 10th to 15th September 1956 at Government Offices, Horse Guards Avenue, London, S.W.1.

Those wishing to participate in the Congress, either by contributing papers or by supplying exhibits for the exhibition, should apply to:—The Congress Secretariat, Dept. of Scientific and Industrial Research, Charles House, 5-11 Regent Street, London, S.W.1.

Massachusetts Institute of Technology Foreign Student Summer Project 1956

The M.I.T. is offering a small number of Scholarships to university graduates in Science and Technology to enable them to attend the Summer Session of the Institute from 4th June to 25th September 1956. No applications in Architecture and Town Planning can be considered.

Both men and women are eligible for these awards. Candidates must be British citizens under 28 years of age, in good health, who hold recognised university degrees in technical fields and have had at least two years' work in either industrial or academic research since graduating.

Application forms may be obtained after Monday 22nd August 1955 from The Cultural Affairs Officer, American Embassy, Room 302, 41 Grosvenor Square, W. No applications can be considered that are received at the Embassy after Friday 14th October 1955.

Royal Aeronautical Society Staff Requirement

The Society urgently requires an experienced shorthand-typist. Applications should be made to the Secretary giving full particulars of experience.

The Library

The Library has been presented with "Automobilia and Flight," January-July 1909. If any members know of previous or subsequent numbers the librarian will be glad to hear of them. He would also like to have information on "Aeronautical Engineering" which succeeded the American "Aeronautics" circa 1915.

DIARY

BRANCHES

21st **August**
Halton.—Films: Magic in Metal; The Museum of Science. Branch Hut, R.A.F., Halton. 6.45 p.m.

28th August
Halton.—Propellers and the Future. P. Brett. Branch Hut, R.A.F., Halton. 6.45 p.m.

5th **September**
Halton.—The Problems of Interplanetary Flight. Dr. L. R. Shepherd. Branch Hut, R.A.F., Halton. 6.49 p.m.

12th September
Halton.—Films: Argonic Welding and others. Branch Hut, R.A.F., Halton. 6.45 p.m.

14th September
Halton.—Visit to Firth Brown Ltd., Sheffield.

19th September
Halton.—Films: Steelmaking and Toolmaking. Branch Hut, R.A.F., Halton. 6.45 p.m.

21st September
Coventry.—Report by E. D. Keen and Dr. W. F. Hilton on the Fifth Anglo-American Aeronautical Conference. The Wine Lodge, The Burges, Coventry. 7.30 p.m.

23rd September
Birmingham.— **BRANCH AREA MEETING.** Application of a Digital Computor to Gas Turbine Engineering. G. C. Best. Chamber of Commerce, New Street, Birmingham. 7.15 p.m.

26th September
Halton.—Paper by an Aircraft Apprentice. Branch Hut, R.A.F., Halton, 6.45 p.m.

19th October
Coventry.—Cockpit Hoods. Dr. Holland. The Wine Lodge, The Burges, Coventry. 7.30 p.m.

JOURNAL BINDING

Permanent Binding

The new prices for permanent binding of Journals are:

1954 Volume (including packing and postage)	18s. 6d.
1953 Volume (including package and postage)	19s. 0d.
Previous Volumes (including packing and postage)	18s. 0d.

Journals, with a note of the name and address of the sender, should be sent direct to The Lewes Press, Friars Walk, Lewes, Sussex, and the remittance to the Secretary at the Offices of the Society.

Self-Binder Cases

Self-Binder cases of the "Easibind" type to hold 12 Journals (cost 11s. 6d. each) are available from the offices of the Society.

CHANGES OF ADDRESS

To assist in keeping the records of members correct and up to date the Secretary will be glad if all members will notify him as soon as possible of changes of address.

When notifying changes please give the following particulars:—

Name (in block letters). New address (in block letters).
Grade of Membership. Old address.

Changes of address must be received *before the 15th of the month* in order to be effective for the JOURNAL for the following month.

ELECTIONS

The following is a list of new members and transfers of membership of the Society:—

Associate Fellows

Ronald Desmond Anthony (*from Graduate*)
Harold Caplan (*from Student*)
Albert Denton Curry (*from Graduate*)
Norman Aubrey Noel Gillard
William Edrington Gray
Albert Arthur Hawes
Ralph Spencer Hooper (*from Graduate*)
Thomas Henry Kerr (*from Graduate*)
John Robert Linge (*from Graduate*)
Edwin Thomas Marriott
Roger Pierre Mollard
Glenn Murphy
John Radford (*from Graduate*)
Herbert Peter Sullivan
James Walter Useller (*from Graduate*)
Robert George Ward (*from Graduate*)
Edward Wray (*from Graduate*)

Associates

Robert Begg
Eric Lewis Eavis (*from Student*)
Fred Watson Newman (*from Student*)
David William Scott (*from Student*)
Vivian Charles Varcoe

Graduates

Michael Geoffrey Bader
John Robert Bishop
Mahesh Chandra
Philip Morgan Davey (*from Student*)
Alan Johnson
David William McCulloch (*from Student*)
Richard Douglas Cook Passey (*from Student*)
John Lester Rudkin (*from Student*)
Allan Taylor
T. Vargheese Vareed
Alan Leonard Yetteram

Students

Charles Bryant
Dudley Albert Clayton
David George Elms
Robin William Guy
Charles Korte Hickling
Thomas Gerald John Moag
Michael Albert Murden
Iskandar Nahas
Christopher George Plowman
William Morris Salmon
John Foster Wilby
Michael Roy Williams

Companions

Jean Mathieu Hubertus Heines
Margaret Vivienne Lawrence
Raymond Long (*from Student*)
George Henry Wyndham Wharton

REPRINTING ANNUAL REPORTS AND JOURNALS

The Council have agreed that the Annual Reports and the first fifteen Volumes of the *Journal of the Royal Aeronautical Society* be reprinted facsimile, as the stock of these publications is extremely low, and in some cases we have no copies at all. The Council would be very grateful to any member who could lend, or sell to the Society, copies of the issues which are missing so that this reprinting may be done. The missing numbers are:—

Annual Reports	*Journals*	
Nos. 1	Vol. I	1897 April
2	Vol. V	1901 April
4	Vol. V	1901 October
6	Vol. VII	1903 April
9	Vol. VII	1903 July
12	Vol. X	1906 January
18 and 19	Vol. XI	1907 April
20 and 21	Vol. XIII	1909 July
	Vol. XIV	1910 July
	Indexes to Vols. II, IX, XI and XIII	

The Journal of the Royal Aeronautical Society

WITH WHICH IS INCORPORATED THE INSTITUTION OF AERONAUTICAL ENGINEERS

VOLUME 59 AUGUST 1955 NUMBER 536

Definitions of the Thrust of a Jet Engine and of the Internal Drag of a Ducted Body

SUMMARY: The problems which occur in defining the thrust of a jet engine and the internal drag of a ducted body are considered, and formal definitions and names are given for the concepts considered to be of importance.

(Although already published as A.R.C. Current paper No. 190, this paper is considered to be of such general interest that it is published here, at the suggestion of the Aeronautical Research Council, so that it may receive a more permanent form of publication and a wider circulation.—Ed).

1. Introduction

Recent measurements, in flight, of the thrusts of turbo-jet engines[1,2] have emphasised the need for a precise definition of jet-engine thrust, and also of the corresponding drag of the body housing the engine. Such definitions would be of interest to the thermodynamicist (who is concerned with the internal performance of the engine) and to the aerodynamicist (who deals with the air flow external to the engine, in connection with either aircraft performance estimation or thrust measurement in flight). Each of these considers certain forces to be important, and it is clearly necessary that the various terms employed by either should have a precise, unambiguous and mutually-agreed meaning.

The problem of defining the thrust of a jet engine and the drag of the body housing is, of course, only a special case of the general problem of the flow through and round ducted bodies; it is fundamentally the same therefore as the problem of defining the internal and external drag, say, of a piston-engine cooling system, or a ducted heat exchanger. In all such problems it is usually convenient to consider separately the internal and external flows, and the thrusts and drags associated with them. The definitions of these forces must be consistent, however; that is, the definitions must be such that the overall longitudinal force on the body is given by the difference between the total thrust and the total drag. As the manner in which the longitudinal force can be divided into thrust and drag is to some extent arbitrary, it is not surprising that different definitions have been used in different contexts.

Attempts to resolve this problem were made independently by Jakobsson[3] and in an unpublished paper by Warren, who considered what concepts were important and suggested a number of names and precise definitions for the various terms involved. Warren's paper was discussed at a meeting of the Performance Sub-Committee of the Aeronautical Research Council in January 1953, when it was agreed that the subject required further consideration. At the Sub-Committee's request, the Council allowed a temporary Panel to be set up to consider the problem of defining thrust and drag, and also the related problem of the definitions to be used in the description and analysis of the drag of an aircraft. The Panel, which met first on 1st May 1953, was composed of the following members, who were nominated either by the Performance or by the Engine Aerodynamics Sub-Committees.

NOMINATED BY THE PERFORMANCE SUB-COMMITTEE

Professor W. A. Mair (Cambridge University), Chairman
Mr. D. W. Bottle (A. & A.E.E.)
Dr. R. C. Pankhurst (N.P.L.)
Mr. T. V. Somerville (R.A.E.)
Mr. C. O. Vernon (S.B.A.C.)
Mr. C. H. E. Warren (R.A.E.)
Mr. E. W. E. Rogers (N.P.L.), Secretary.

NOMINATED BY THE ENGINE AERODYNAMICS SUB-COMMITTEE

Mr. S. Gray (Eng., R(a)M.O.S.)
Mr. H. Pearson (S.B.A.C.)
Dr. J. Seddon (R.A.E.)
Mr. A. R. Howell (N.G.T.E.) attended some of the meetings.

The present Report deals with the Panel's investigation of the problem of defining jet engine thrusts and concepts related to this; a Report dealing with the analysis of drag will be issued separately. The present investigation is limited to the case where the thrust acts in the direction of the undisturbed stream (*i.e.* along the line of flight), the jet exit plane being normal to this direction. The more complicated cases of an inclined thrust axis, an oblique jet exit plane*, or a jet exit plane upstream of the engine cowling exit with internal mixing between cooling and jet flows, have been borne in mind throughout the Panel's discussions, but it was felt that it would be better to defer consideration of these more complicated problems until agreement had been reached for the simpler case.

During the investigation it became clear that it might be convenient to introduce names for certain quantities having the dimensions of pressure. The Panel found, when reviewing these quantities, that some of the definitions relating to pressure, as laid down in the British Standard Glossary of Aeronautical Terms[4],

*If the nozzle is not square-cut and there is still some convergence in the final part of the nozzle, it is clear that some deflection of the issuing jet from the free-stream direction will occur, entailing certain fundamental difficulties. Even if there is no convergence, some deflection will probably occur when the jet is choked.

were wholly unacceptable for compressible flow. The Panel would have liked to have suggested more suitable definitions, but was unable to agree on the names for certain quantities, or even on the need for names in certain cases†.

Notation

The symbols in this list have been used in writing this report, but it is not recommended that they should all necessarily be adopted for general use.

- A station area (normal to the direction of the undisturbed stream)
- A_f effective nozzle area quoted by engine manufacturers
- D_B internal drag according to Betz's formula (*see* equation (25))
- D_J internal drag according to Jones' formula (*see* equation (24))
- M Mach number, given by $V/\sqrt{(\gamma p/\rho)}$
- m rate of mass-flow
- P static pressure
- P_b local static pressure of the external fluid surrounding the jet at exit
- P_∞ pressure of the undisturbed stream
- V resultant velocity
- γ ratio of specific heats of fluid
- Θ thrust, given by equation (1)
- Θ_{int} intrinsic thrust (*see* equation (4))
- Θ_{Jg} gross Jones thrust, Θ_{Jn} net Jones thrust (*see* equation (8))
- Θ_{Pg} gross Pearson thrust, Θ_{Pn} net Pearson thrust (*see* equations (10(*a*) and (*b*))
- Θ_{post} post-exit thrust (*see* equation (5))
- Θ_{pre} pre-entry thrust (*see* equation (3))
- Θ_{sg} gross standard thrust, Θ_{sn} net standard thrust (*see* equations (6) and (7))
- ρ fluid density
- ϕ angle between local direction of flow and direction of undisturbed stream (positive when streamtube is diverging)

Suffices

- e denotes the exit station (*see* Fig. 1)
- f denotes the station, corresponding to the "effective area" of the jet, where the flow is parallel to the undisturbed stream (*see* Appendix III)
- g denotes the station at which the static pressure of the internal fluid attains the local static pressure of the external fluid
- i denotes the intake station (*see* Fig. 1)
- w denotes a station at infinity downstream, at which the internal static pressure is uniform and equal to the static pressure of the undisturbed stream (*see* Fig. 1)
- ∞ denotes a station at infinity upstream (*see* Fig. 1)

†In any case, a revision of the names for various concepts relating to pressure is outside the terms of reference of the Panel, and is doubtless a matter for a more widely representative body than one set up primarily to consider the definitions of thrust and drag.

2. The Problem

The resultant force on a stationary ducted body in a steady stream of infinite extent can be obtained by considering the change of momentum of the fluid contained within an arbitrary surface completely surrounding the body. It is desirable, however, to divide the resultant force into internal "thrust" and external "drag" forces, the general case being considered as one in which the internal flow contributes a thrust rather than a drag, in view of the importance of jet engines. The results obtained will be immediately applicable to the related problem of internal drag, simply by interpreting thrust as negative drag. The internal thrust and external drag could be defined as the sums of the components in the appropriate directions of the aerodynamic forces on the internal and external surfaces of the body, assuming that these surfaces could be precisely defined. Such an analysis, although quite valid, would be of interest mainly to the structural worker, who is concerned with the forces on the various parts of the body itself. It would be of less value, as will be seen later, to the aerodynamicist and thermodynamicist, who are concerned mainly with the fluid and the changes in its state, and who require definitions of the forces in terms of the rate of change of momentum of the fluid that flows through the body (internal flow) and of the fluid that flows round it (external flow).

In considering changes of momentum in the fluid flowing past a ducted body, it has sometimes not been clear whether the relevant static pressures should be measured absolutely or relative to some datum pressure. For a closed (unducted) body this difficulty does not arise, since the subtraction of a datum pressure is equivalent to integrating a constant pressure over a closed surface and does not contribute to the overall force. But in considering *parts* of the surface and of the flow, as is done with ducted bodies, the derived forces depend on the datum pressure used. For a body at rest relative to the surrounding fluid the undisturbed fluid pressure acts on all surfaces, and the resultant force is zero. When there is relative motion between the fluid and the body this equilibrium is disturbed and the surface forces which are of significance are those due to the changes in surface pressure arising from the motion. Thus in the derivation of these forces all pressures must be referred to the undisturbed static pressure as a datum. This practice is adopted in the present Report.

The question of the division of the resultant force into internal "thrust" and external "drag" components resolves into a choice of the upstream and downstream stations between which the momentum relations are applied.

If the reference stations are taken at the entry and exit, the calculated values of these components will in general vary with any change in entry or exit area, although there would not necessarily be any change in the overall flow pattern. Thus if the thrust were defined in this way, the engine manufacturer would not be able to express the performance of his engine by a compact series of curves, applicable to all installations. Moreover, he would not be able to measure the thrust with-

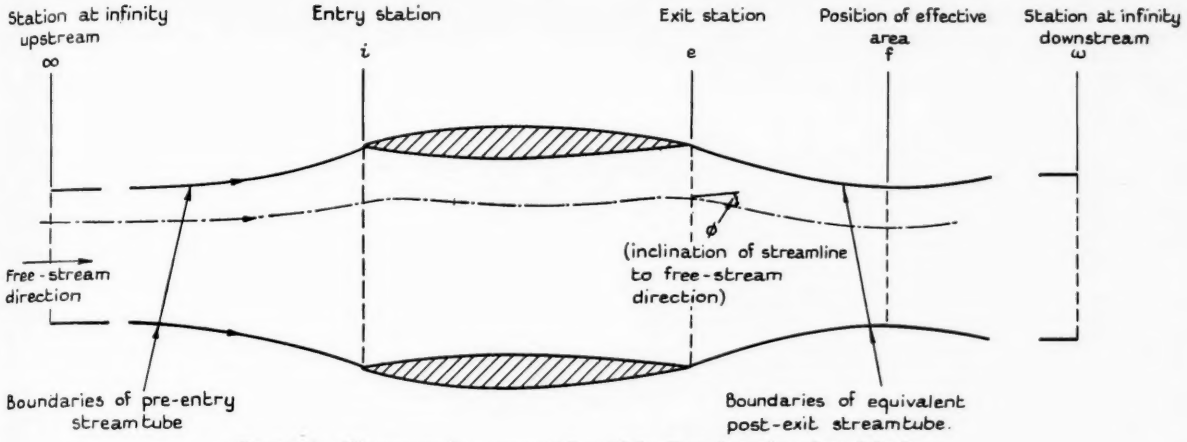


FIGURE 1. Diagrammatic representation of the flow through a ducted body.

out traversing the flow at both the entry and exit stations. There would also be difficulty in defining the location of the entry station precisely. The use of the entry and exit stations is therefore particularly unsuitable for the definition of thrust, and some more generally applicable definition is required.

If a new reference station is considered, upstream of the entry station, then the calculated force associated with the internal flow will differ from that actually exerted on the internal surfaces of the body. It is shown in Appendix II that this difference is equal to the longitudinal component of the pressure forces acting on the boundaries of the "pre-entry stream tube" (Fig. 1). A corresponding treatment of the external flow would yield a force differing from that actually exerted on the external surfaces by an equal and opposite amount. Physically it can be said that the flow in the pre-entry stream tube, induced by the engine, influences the external flow and thereby the pressure distribution on the external surfaces. Thus a force, equal in magnitude to the longitudinal component of the pressure forces on the boundaries of the pre-entry stream tube, is rightly attributed to the engine thrust, though usually acting on the external surfaces. This is in fact the basis on which engine thrusts have usually been derived in the past. Treating the external forces as positive in the drag sense, then a corresponding external drag is obtained which is equal to the force actually exerted on the external surfaces (as would be derived from the actual pressure distribution) *increased* by a quantity which has sometimes been referred to as the "pre-entry drag."

A similar argument can be applied to the flow downstream of the exit station by introducing the concept of a "post-exit stream tube" separating the internal and external flows. It is concluded that the fundamental definition of the thrust of a jet engine should be framed in terms of the stations at infinity upstream and infinity downstream.

Consideration of the pre-entry flow presents no difficulties, but there is in fact no physical post-exit stream tube, the flow downstream of the exit being complicated by the transfer of momentum and energy between the internal and external flows in the wake of the body. It has been customary in the past to assume that the local static pressure in the external flow in the region of the jet exit is equal to the undisturbed static pressure. In general, these pressures are not equal and it is necessary to take account of this in framing definitions of thrust. Moreover, the pressure of the internal fluid at the exit station may differ from that in the adjacent external fluid, and shock waves may be present in either the internal or external flows, or both. A major problem, therefore, is to devise practical definitions of thrust which take account of the complex flow conditions downstream of the exit, and which enable the thrust to be determined from practicable measurements. This is considered in the following Sections.

Some cautionary remarks about central bodies and side intakes are given in Appendix IV.

3. The Thrust and Related Quantities

The thrust obtained as a result of choosing reference stations at infinity both upstream and downstream of the body, and by assuming the existence of an effective post-exit stream tube, is given by the equation (derived in Appendix II)

$$\Theta = \int \rho_w V_w^2 dA_w - \rho_\infty V_\infty^2 A_\infty, \quad . \quad (1)$$

where A is the area of the station (normal to the free-stream direction), ρ is the density of the fluid, and V is the local velocity, parallel to the free stream at both stations. Suffix "w" refers to the station at infinity downstream (where the static pressure is uniform and equal to that of the undisturbed stream), suffix "∞" refers to conditions at infinity upstream, and the integration is made over the mass of fluid flowing through the body.

The thrust given by equation (1) may be divided into three components:

$$\Theta = \Theta_{pre} + \Theta_{int} + \Theta_{post}, \quad . \quad . \quad (2)$$

where Θ_{pre} and Θ_{post} are the *pre-entry* and *post-exit*

thrusts, associated with the pressure exterted by the internal flow upon the external flow ahead of and behind the body, and Θ_{int} is the *intrinsic thrust**, associated with the force on the internal surfaces of the body.

The components of Θ may be written as

$$\Theta_{pre} = \int (\rho_i V_i^2 \cos^2 \phi_i + P_i - P_\infty)\, dA_i - \rho_\infty V_\infty^2 A_\infty, \quad (3)$$

$$\Theta_{int} = \int (\rho_e V_e^2 \cos^2 \phi_e + P_e - P_\infty)\, dA_e - \int (\rho_i V_i^2 \cos^2 \phi_i + P_i - P_\infty)\, dA_i, \quad (4)$$

$$\Theta_{post} = \int \rho_w V_w^2 dA_w - \int (\rho_e V_e^2 \cos^2 \phi_e + P_e - P_\infty)\, dA_e, \quad (5)$$

where suffices "i" and "e" denote the intake and exit stations respectively, and ϕ is the local inclination of a streamline to the free-stream direction.

The term

$$\rho_\infty V_\infty^2 A_\infty$$

in equation (3) is the "ram drag"†; by continuity this is equal to $m_i V_\infty$, where m_i is the internal rate of mass-flow at the duct entry.

The pre-entry thrust depends on the conditions at the entry and in the undisturbed stream, which are assumed to be known. The intrinsic thrust depends in addition on the conditions at exit, which in principle are assumed to be measurable or calculable. It follows from equation (2) that the use of Θ depends on the successful determination of the post-exit thrust, and thus in any practical definitions of thrust, an attempt must be made to relate the conditions at infinity downstream to those at the jet exit; this approach is considered in Section 4.

The presence of a post-exit thrust depends on the existence, in the region surrounding the jet exit, of pressures differing from that of the undisturbed stream, and the value of the post-exit thrust depends on the magnitudes of these pressures and the subsequent behaviour of the combined internal and external flows. Hence any assessment of the post-exit thrust must rely on a measured, or assumed, distribution of pressure in the region surrounding the jet at exit. (The presence of a post-exit thrust implies that the internal flow from the exit affects the pressure distribution on the external surfaces.)

Some consideration must also be given to the conditions of the issuing jet at exit. When measurements are made at this position in flight or on a test-bed, the magnitude and direction of the resultant velocity must be found, at all points of the cross section, in order to obtain the velocity component $V_e \cos \phi_e$. The engine manufacturer, in preparing brochures from which estimates can subsequently be made of the engine thrust in given conditions, envisages a somewhat idealised picture of the jet flow at the exit and assumes that the gas velocity is entirely in the direction of the undisturbed stream. An "effective" exit area A_f (at some station "f") is then quoted by the manufacturer, which differs from the real exit area A_e, to compensate for this simplification. These two viewpoints will be borne in mind when deriving expressions for thrust to be used in practice. A further discussion of the relation between quantities at stations "e" and "f" is given in Appendix III.

4. Practical Definitions of Thrust*

4.1. THE STANDARD THRUST

If the post-exit thrust can be neglected, the thrust is given by the sum of the pre-entry and intrinsic thrusts. This is called the *net standard thrust* and is the thrust normally specified in engine brochures. In terms of the conditions at the duct exit station the net standard thrust is defined by

$$\Theta_{sn} = \int (\rho_e V_e^2 \cos^2 \phi_e + P_e - P_\infty)\, dA_e - m_i V_\infty. \quad (6)$$

When an effective area is given (at station "f"), the net standard thrust is

$$\Theta_{sn} = \int (\rho_f V_f^2 + P_f - P_\infty)\, dA_f - m_i V_\infty. \quad (7)$$

In equation (7), $\cos^2 \phi$ does not appear because at station "f" the flow is assumed to be in the direction of the undisturbed stream. The omission of the second term (the ram drag) in either of the above equations gives the *gross standard thrust* (Θ_{sg}).

4.2. THE JONES THRUST

If the jet is subsonic at exit, and exhausts into a region where the pressure differs from P_∞, the simplest assumption that can be made for calculating the post-exit thrust is that there is no transfer of momentum or energy, during the subsequent flow to infinity where the pressure is P_∞. It is recommended that the thrust defined by this model of the flow, in which the fluid downstream of the exit is assumed to be expanded or compressed adiabatically and isentropically to the undisturbed pressure P_∞, should be called the *net Jones* thrust, since it represents an extension of B. M. Jones' original method[5] of obtaining the drag of a body in incompressible flow, to the problem of determining thrust.

Using the condition of continuity of mass-flow in the wake and the energy equation for the compressible flow of a perfect gas, it may be shown (Appendix II) that the

*The term "intrinsic thrust" is preferred to "internal thrust" because throughout this Report "internal" means "relating to the fluid that flows through the body," which embraces the fluid ahead of and behind the body as well as that actually within the body. The word "intrinsic" is thought to be etymologically suitable in this context.

†The term "ram drag" is due to Jakobssen[3] and is to be preferred to the term "intake drag," which is also used to denote the aerodynamic drag of the actual intake. Ram drag is the same as the "sink drag" which occurs with suction aerofoils.

*See Appendix II for a more detailed development of the equations given in this Section.

net Jones thrust is given, in terms of conditions at the exit plane, by

$$\Theta_{Jn}=\int \rho_e V_e^2\left[1+\frac{2}{(\gamma_e-1)M_e^2}\times\right.$$
$$\left.\times\left\{1-\left[\frac{P_\infty}{P_e}\right]^{\frac{\gamma_e-1}{\gamma_e}}\right\}\right]^{\frac{1}{2}}\cos\phi_e\,dA_e-m_i V_\infty. \quad (8)$$

The ratio of the specific heats of the fluid at exit (γ_e) is assumed to remain constant in the subsequent flow to infinity, but may differ from the value of γ appropriate to conditions ahead of the body.

Substituting suffix "f" for suffix "e" and putting $\cos\phi$ equal to unity in equation (8) would give the net Jones thrust in terms of conditions at the idealised flow plane "f," where the flow is assumed to be everywhere parallel to the undisturbed stream.

As before, the omission of the term $m_i V_\infty$ leads to the *gross Jones thrust* (Θ_{Jg}).

4.3. THE PEARSON THRUST

When there is a difference of pressure across the solid boundary between the internal and external flows at an infinitesimal distance upstream of the exit station, as there may be when the jet is supersonic, a more realistic assumption is that the issuing fluid either expands or is compressed until it attains the local static pressure of the surrounding fluid P_b (assumed uniform), by an adiabatic process in which the quantity

$$(\rho V^2\cos^2\phi+P-P_b)\,dA \quad (9)$$

remains constant along each streamtube. It is assumed that when the pressures in the internal and external flows have become equal, the flow is parallel to the direction of the undisturbed stream (station "g") and there is then no transfer of momentum or energy during the subsequent flow to infinity where the pressure is P_∞.

Applying the energy equation to both parts of the flow (between the jet exit and station "g," and between station "g" and infinity downstream) and using the condition that the quantity given by (9) remains constant between the exit and station "g," the thrust obtained is (*see* Appendix II)

$$\Theta_{Pn}=\int \rho_e V_e^2\left[\left[\frac{\rho_e V_e^2\cos^2\phi_e+P_e-P_b}{\rho_e V_e^2}\right]^2\left(\frac{P_\infty}{P_b}\right)^{\frac{\gamma_e-1}{\gamma_e}}+\right.$$
$$+\left\{1-\left(\frac{P_\infty}{P_b}\right)^{\frac{\gamma_e-1}{\gamma_e}}\right\}\left\{\frac{2}{(\gamma_e-1)M_e^2}+1\right\}$$
$$\left.\cos^2\phi_e\right]^{\frac{1}{2}}dA_e-m_i V_\infty. \quad (10a)$$

Alternatively, in terms of conditions at the idealised flow station "f," this can be written

$$\Theta_{Pn}=\int \rho_f V_f^2\left[\left[\frac{\rho_f V_f^2+P_f-P_b}{\rho_f V_f^2}\right]^2\left(\frac{P_\infty}{P_b}\right)^{\frac{\gamma_f-1}{\gamma_f}}+\right.$$
$$\left.+\left\{1-\left(\frac{P_\infty}{P_b}\right)^{\frac{\gamma_f-1}{\gamma_f}}\right\}\left[\frac{2}{(\gamma_f-1)M_f^2}+1\right]\right]^{\frac{1}{2}}dA_f-m_i V_\infty. \quad (10b)$$

It is recommended that the thrust Θ_{Pn} be called the *net Pearson thrust*, after the member of the Panel who derived the equations; the *gross Pearson thrust* Θ_{Pg} results from the exclusion of the ram-drag component $m_i V_\infty$.

5. Comparison of the Three Practical Definitions of Thrust

If the jet discharges into a region where the static pressure P_b is equal to the undisturbed pressure P_∞, the post-exit thrust is zero and the standard thrust is appropriate for all exit velocities and pressures; the Pearson and standard thrusts are then identical. The former may therefore be regarded as a generalisation of the standard thrust, to be used when the jet exhausts into a region whose local static pressure differs from that of the free stream.

Where P_b is not equal to P_∞, there may be a post-exit thrust. For a subsonic jet, there can be no difference of pressure across the boundary between the internal and external flows at the exit, and if the flow inclination at the exit is small, so that $\cos\phi_e\approx 1$, the Pearson and Jones thrusts are then equal. Hence the Pearson thrust can be regarded as a generalisation of the Jones thrust to the case where a difference of static pressure exists between the internal and external flow at exit, as in a choked or supersonic jet.

Finally, a comparison may be made of the Jones and standard thrusts. For this purpose the flow inclination at exit ϕ_e will again be assumed to be small, so that $\cos\phi_e=1$. Expanding the radical in equation (8) in powers of $(P_e-P_\infty)/\rho_e V_e^2$ then gives*

$$\Theta_{Jg}=\int \rho_e V_e^2\left[1+\frac{P_e-P_\infty}{\rho_e V_e^2}+\right.$$
$$\left.+\frac{M_e^2-1}{2}\left(\frac{P_e-P_\infty}{\rho_e V_e^2}\right)^2+\ldots\ldots\right]dA_e$$

whereas, with the same assumption that $\cos\phi_e=1$,

$$\Theta_{sg}=\int \rho_e V_e^2\left[1+\frac{P_e-P_\infty}{\rho_e V_e^2}\right]dA_e.$$

Thus (if ϕ_e is small) the standard thrust is equal to the first two terms only of the expanded formula for the Jones thrust. The difference between the two thrusts is of course equal to the post-exit thrust, and is shown in Fig. 2.

The condition of zero momentum or energy transfer in the post-exit flow, assumed in deriving the Jones thrust, will not be obtained in practice, although with a subsonic jet the errors involved are unlikely to be large. In certain cases, however, the use of the Jones thrust is clearly wrong. For example, when the total pressure of the jet at exit is less than the static pressure of the undisturbed stream, the fluid flowing through the body cannot attain the pressure of the undisturbed stream again, unless there is some transfer of momentum or energy† Alternatively, if the velocity at exit is sonic

*The subsequent equations could equally well be written in terms of conditions at station "f."

†This type of flow may occur for example in the flow through a piston-engine cooling duct as the exit flap is opened (*see also* Appendix V).

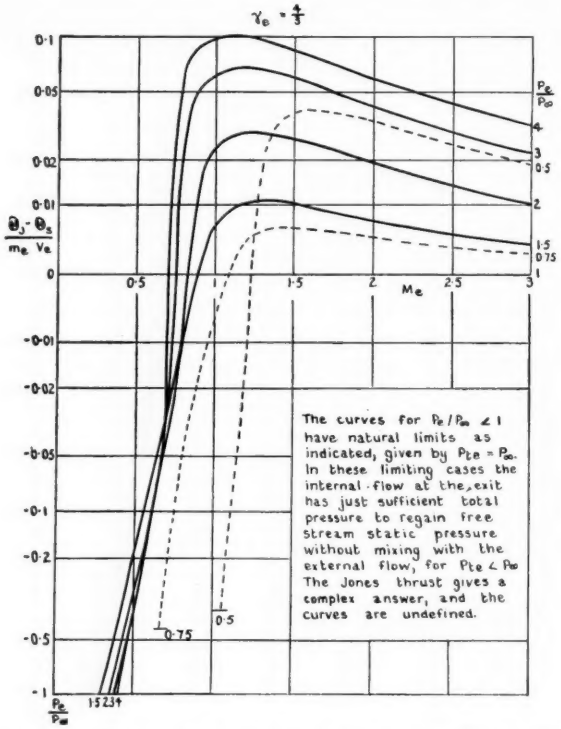


FIGURE 2. Curves of the difference between the Jones thrust and the standard thrust.

or supersonic, there can be a difference of static pressure at exit between the internal and external flows. With an ideal fluid and no mixing, the wake could exhibit a periodic flow pattern extending to infinity and there would be no station at which the internal static pressure was at once uniform and equal to that of the undisturbed stream.

The latter case, which is the more important in thrust problems, can be dealt with by using the Pearson thrust. The assumption made in deriving equations 10 (*a*) and (*b*) probably represent the best simple model of the flow that it is possible to obtain, though this model may not always correspond with reality. If, after the initial non-isentropic process, there is substantial mixing of the internal and external flows before the jet has reached the pressure P_∞, the actual thrust obtained will be affected and the assumption made by Pearson violated. In addition the external flow itself may not be isentropic; for example shock waves may be present. Moreover the Pearson thrust depends on the value of P_b; in practice this is unlikely to be uniform and some mean pressure must be used in the calculations. It is felt, however, that points such as these do not seriously affect the usefulness of the flow model and resulting thrust given by Pearson.

6. Internal Drag and Propulsive Thrust

In problems of internal drag, similar concepts to the three types of thrust given in Section 4 are required. It is therefore recommended that the quantities defined by the negative of the three types of net thrust should be called the *standard internal drag*, the *Jones internal drag* and the *Pearson internal drag*, if such quantities are required. The adjective net is unnecessary because the concept of gross internal drag has no significance. Some formulae for internal drag are given in Appendix V.

The overall longitudinal force on a nacelle housing an engine, treated as a whole, is felt to be worthy of a name and it is suggested that this be called the *propulsive thrust*, as it is the thrust available for propelling the rest of the aircraft.

7. Concluding Remarks and Recommendations

The foregoing investigation has shown that the thrust can be divided into pre-entry thrust, intrinsic thrust and post-exit thrust. In the absence of the last component, the standard thrust is obtained. Two idealised models of the flow downstream from the exit were put forward and enabled an estimate to be made of the post-exit thrust in certain cases. This led to the concepts of Jones thrust and Pearson thrust, which can be used to assess the amount of overall longitudinal force that it is fair to attribute to the internal flow. The flow models are not claimed to be exact representations of the actual flow but it is felt that they are sufficiently realistic to be worth serious consideration. At this stage it is not possible to estimate with any accuracy their possible usefulness. There are related internal drag forces corresponding to each of the three types of thrust.

It is suggested that, of the thrusts considered here, the Pearson thrust represents the best measure of the thrust that it is fair to ascribe to a jet engine in the general case. When the internal static pressure at the exit station is equal to the local static pressure surrounding the jet exit (subsonic, or unchoked, jet), the Pearson thrust reduces to the Jones thrust. When the local static pressure surrounding the jet exit is equal to the undisturbed static pressure, the Pearson thrust reduces to the standard thrust; in this case the post-exit thrust is zero and the standard thrust is the correct measure of the thrust.

For ease of reference the various thrusts are collated and briefly defined in Appendix I.

APPENDIX I

DEFINITIONS OF SOME QUANTITIES USED IN THIS REPORT

1. *Internal Flow.*—The flow of fluid which passes through the duct.

2. *External Flow.*—The flow of fluid which passes around the outside of the duct.

3. *Pre-entry Stream Tube.*—The stream tube, extending from infinity upstream of the body to the duct entry, whose boundaries separate the internal and external flows.

4. *Equivalent Post-exit Stream Tube.*—The stream tube, extending from the duct exit to infinity downstream, whose boundaries are assumed to separate the internal and external flows in the absence of mixing.

5. *Ram Drag.*—The force, acting in the downstream direction which would arise if the momentum of the internal flow at infinity ahead of the body were destroyed. (It is equal to the product of the rate of mass flow at the duct entry and the velocity of the undisturbed relative airstream, and is the effective sink drag of the air intake.)

6. *Thrust* (associated with the internal flow).—The force corresponding to the rate of increase of momentum, in a direction parallel to the undisturbed relative airstream, of the internal flow through the duct, this increase being calculated between stations at infinite distances upstream and downstream of the body. The thrust can be divided conveniently into three parts, the pre-entry thrust, the intrinsic thrust, and the post-exit thrust.

7. *Pre-entry Thrust.*—The force, in a direction parallel to the undisturbed relative airstream, arising from the pressure forces acting on the internal surface of the pre-entry stream tube.

8. *Intrinsic Thrust.*—The force, in a direction parallel to the undisturbed relative airstream, arising from the pressure and friction forces acting on the internal surface of the duct.

9. *Post-exit Thrust.*—The force, in a direction parallel to the undisturbed relative airstream, arising from the pressure forces acting on the internal surface of the equivalent post-exit stream tube.

10. *Standard Thrust.*—The net standard thrust is the sum of the pre-entry thrust and the intrinsic thrust. The gross standard thrust is the arithmetic sum of the net standard thrust and the ram drag.

11. *Jones Thrust.*—The net Jones thrust is the sum of the pre-entry thrust, the intrinsic thrust and the post-exit thrust, the last being calculated by assuming that the fluid that has passed through the duct is compressed or expanded adiabatically and isentropically downstream of the duct exit until the undisturbed static pressure of the relative airstream is attained. The gross Jones thrust is the arithmetic sum of the net Jones thrust and the ram drag.

12. *Pearson Thrust.*—The net Pearson thrust is the sum of the pre-entry thrust, the intrinsic thrust and the post-exit thrust, the last being calculated by assuming that the flow downstream of the duct exit has two stages. In the first, the fluid is compressed or expanded adiabatically and irreversibly until the local static pressure of the surrounding fluid (assumed uniform) is attained, the quantity $(\rho V^2 \cos^2 \phi + P - P_b)\, dA$ remaining constant along each streamline; the flow is then assumed to be parallel to the direction of the undisturbed relative airstream. In the second stage the fluid is compressed or expanded adiabatically and isentropically until the undisturbed static pressure of the relative airstream is attained. The gross Pearson thrust is the arithmetic sum of the net Pearson thrust and the ram drag.

13. *Propulsive Thrust.*—The overall force exerted on a nacelle housing an engine, in a direction parallel to the undisturbed relative airstream; it is considered positive when acting in the upstream direction.

14. *Standard Internal Drag.*—The negative of the net standard thrust.

15. *Jones Internal Drag.*—The negative of the net Jones thrust.

16. *Pearson Internal Drag.*—The negative of the net Pearson thrust.

APPENDIX II

DERIVATION OF THE THRUST AND RELATED QUANTITIES

Figure 1 represents the "internal" flow ahead of, through and to the rear of a stationary ducted body. Consider the control volume bounded by the intake and exit stations "i" and "e" (taken for convenience as normal to the free-stream direction) and the internal surfaces of the body. Momentum considerations show that the sum of the pressure and friction forces acting on the fluid at the boundaries is equal to the rate of change of momentum of the fluid flowing through the control volume. Resolving in the direction of the free stream gives

$$\int_i P_i\, dA_i - \int_e P_e\, dA_e + \int_{int} (P \sin\phi - F\cos\phi)\, dS = \int_e \rho_e V_e^2 \cos^2\phi_e\, dA_e - \int_i \rho_i V_i^2 \cos^2\phi_i\, dA_i,$$

where F is the local friction force per unit area, dS is an element of area of the internal surface, and ϕ is considered to be positive when the duct is diverging. (The third integral on the left-hand side represents the force in the free-stream direction exerted on the fluid by the internal surfaces of the duct.)

Hence $$\int_{int} (P\sin\phi - F\cos\phi)\, dS = \int_e (\rho_e V_e^2 \cos^2\phi_e + P_e)\, dA_e - \int_i (\rho_i V_i^2 \cos^2\phi_i + P_i)\, dA_i.$$

The integration of a constant pressure P_∞ (the undisturbed static pressure) round the closed boundary must give zero force,

i.e. $$\int_i P_\infty dA_i - \int_e P_\infty dA_e + \int_{int} P_\infty \sin\phi\, dS = 0.$$

Hence by subtraction

$$\int_{int} [(P - P_\infty)\sin\phi - F\cos\phi]\, dS = \int_e (\rho_e V_e^2 \cos^2\phi_e + P_e - P_\infty)\, dA_e - \int_i (\rho_i V_i^2 \cos^2\phi_i + P_i - P_\infty)\, dA_i.$$

The forces that are of significance are those associated with pressures measured relative to the undisturbed static pressure. Thus the left-hand side of the above equation gives the force in the free-stream direction exerted on the fluid by the internal surfaces of the duct, measured positively in the downstream direction. This is clearly equal to the effective force in the direction of the free-stream exerted by the fluid on the internal surfaces, measured positively in the *upstream* (*thrust*) direction, and this quantity will be defined as the "*Intrinsic Thrust*," Θ_{int}. Thus

$$\Theta_{int} = \int_e (\rho_e V_e^2 \cos^2\phi_e + P_e - P_\infty)\, dA_e - \int_i (\rho_i V_i^2 \cos^2\phi_i + P_i - P_\infty)\, dA_i. \quad . \quad (11)$$

Now consider the flow between stations at infinity upstream (∞) and the exit (e); in similar manner a "thrust" can be derived which is given by

$$\int_e (\rho_e V_e^2 \cos^2\phi_e + P_e - P_\infty)\, dA_e - \int_\infty (\rho_\infty V_\infty^2 + P_\infty - P_\infty)\, dA_\infty, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and is equal to the sum of the longitudinal forces on the boundaries of the pre-entry stream tube and the internal surfaces.

By subtraction, then: —

("Thrust" given by eqn. (12)) $-\Theta_{int}$ = axial force on pre-entry stream tube

$$= \int_i (\rho_i V_i^2 \cos^2 \phi_i + P_i - P_\infty)\, dA_i - \int_\infty \rho_\infty V_\infty^2\, dA_\infty.$$

This will be defined as the "*Pre-entry Thrust*," Θ_{pre}.

Thus

$$\Theta_{pre} = \int_i (\rho_i V_i^2 \cos^2 \phi_i + P_i - P_\infty)\, dA_i - m_i V_\infty, \qquad (13)$$

since V_∞ is constant over the whole area A_∞ and $m_i = \rho_\infty V_\infty A_\infty$.

Similarly, if the flow is considered between stations at the exit and at infinity downstream (w), a thrust is introduced which is defined as the "*Post-exit Thrust*" and given by

$$\Theta_{post} = \int_w \rho_w V_w^2\, dA_w - \int (\rho_e V_e^2 \cos^2 \phi_e + P_e - P_\infty)\, dA_e, \qquad (14)$$

since $P_w = P_\infty$ and $\phi_w = 0$, although V_w is not uniform.

Since both the pre-entry and post-exit thrusts are communicated to the body, the total thrust is given by

$$\Theta = \Theta_{pre} + \Theta_{int} + \Theta_{post}$$
$$= \int \rho_w V_w^2\, dA_w - m_i V_\infty.$$

The *net standard thrust* is defined as the sum of the pre-entry and intrinsic thrusts and is given by

$$\Theta_{sn} = \int (\rho_e V_e^2 \cos^2 \phi_e + P_e - P_\infty)\, dA_e - m_i V_\infty, \qquad (16)$$

or a similar equation in terms of station "f."

In the Jones model of the flow downstream from station "e" or "f," the expansion or compression from the local pressure P_b to the pressure of the undisturbed stream P_∞ at station "w" is adiabatic and isentropic, no mixing occurring between the external and internal flows.

The energy equation for the compressible flow of a perfect gas can be written

$$\frac{\gamma_e}{\gamma_e - 1}\frac{P_e}{\rho_e} + \tfrac{1}{2}V_e^2 = \frac{\gamma_e}{\gamma_e - 1}\frac{P_w}{\rho_w} + \tfrac{1}{2}V_w^2. \qquad (17)$$

Using the condition of isentropic flow, this equation becomes

$$V_w = V_e\left[1 + \frac{2}{(\gamma_e - 1)M_e^2}\left\{1 - \left(\frac{P_\infty}{P_e}\right)^{\frac{\gamma_e - 1}{\gamma_e}}\right\}\right]^{\frac{1}{2}}.$$

Substituting this value of V_w into equation (15) and using the condition of continuity,

$$\rho_w V_w dA_w = \rho_e V_e \cos \phi_e dA_e$$

gives the *net Jones thrust*

$$\Theta_{Jn} = \int \rho_e V_e^2 \left[1 + \frac{2}{(\gamma_e - 1)M_e^2}\left\{1 - \left(\frac{P_\infty}{P_e}\right)^{\frac{\gamma_e - 1}{\gamma_e}}\right\}\right]^{\frac{1}{2}} \cos \phi_e dA_e - m_i V_\infty. \qquad (18a)$$

If the station "f" is used, ϕ_f is zero and

$$\Theta_{Jn} = \int \rho_f V_f^2 \left[1 + \frac{2}{(\gamma_f - 1)M_f^2}\left\{1 - \left(\frac{P_\infty}{P_f}\right)^{\frac{\gamma_f - 1}{\gamma_f}}\right\}\right]^{\frac{1}{2}} dA_f - m_i V_\infty. \qquad (18b)$$

The Pearson model of the flow has for its initial stage a process in which the fluid flowing internally expands or contracts adiabatically and irreversibly from the internal static pressure at the exit to the local external pressure P_b, the process being complete at station "g" where the flow is assumed to be parallel to the undisturbed stream. During this process, the pressure at the dividing surface between the internal and external flows is assumed to have the uniform value P_b, and for each stream tube the quantity $(\rho V^2 \cos^2 \phi + P - P_b)\, dA$ is assumed to remain constant between the exit and station "g."

The gross Pearson thrust Θ_{Pg} is given by the general equation

$$\Theta = \int_w \rho_w V_w^2 dA_w$$

with conditions at station "w" related to those at the exit by means of the flow model suggested by Pearson. By continuity,

$$\rho_w V_w dA_w = \rho_e V_e \cos \phi_e dA_e.$$

Thus
$$\Theta_{Pg} = \int \rho_e V_e V_w \cos \phi_e dA_e. \qquad (19)$$

For the isentropic and adiabatic flow between stations "g" and "w," the energy equation for a compressible fluid can be used, as in considering the Jones thrust, to give

$$V_w = V_g\left[\frac{2}{(\gamma_e - 1)M_g^2}\left\{1 - \left(\frac{P_\infty}{P_b}\right)^{\frac{\gamma_e - 1}{\gamma_e}}\right\} + 1\right]^{\frac{1}{2}}. \qquad (20)$$

(It is assumed that γ does not change downstream of the exit.) Between stations "e" and "g,"

$$\rho_g V_g^2 dA_g - (\rho_e V_e^2 \cos^2 \phi_e + P_e - P_b)\, dA_e = 0.$$

But by continuity

$$\rho_g V_g dA_g = \rho_e V_e \cos \phi_e dA_e,$$

and hence
$$V_g = \frac{\rho_e V_e^2 \cos^2 \phi_e + P_e - P_b}{\rho_e V_e \cos \phi_e}. \qquad (21)$$

It remains now to eliminate M_g from equation (20). This can be done by applying the energy equation to the adiabatic flow between "e" and "g," giving the following equation

$$\frac{2}{(\gamma_e - 1)M_g^2} = \left(\frac{V_e}{V_g}\right)^2 \left\{\frac{2}{(\gamma_e - 1)M_e^2} + 1\right\} - 1. \qquad (22)$$

Substituting (21) and (22) in (20) and simplifying the result, leads to the following expression for the *net Pearson thrust* when the ram drag term is included

$$\Theta_{Pn} = \int \rho_e V_e^2 \left[\left[\frac{\rho_e V_e^2 \cos^2 \phi_e + P_e - P_b}{\rho_e V_e^2}\right]^2 \left(\frac{P_\infty}{P_b}\right)^{\frac{\gamma_e - 1}{\gamma_e}} + \left[1 - \left(\frac{P_\infty}{P_b}\right)^{\frac{\gamma_e - 1}{\gamma_e}}\right]\left[\frac{2}{(\gamma_e - 1)M_e^2} + 1\right]\cos^2 \phi_e\right]^{\frac{1}{2}} dA_e - m_i V_\infty \qquad (23a)$$

In terms of conditions at station "f," where the flow is assumed to be parallel to the direction of the free stream (*i.e.* $\phi_f = 0$) the net Pearson thrust is

$$\Theta_{Pn} = \int \rho_f V_f^2 \left[\left[\frac{\rho_f V_f^2 + P_f - P_b}{\rho_f V_f^2}\right]^2 \left(\frac{P_\infty}{P_b}\right)^{\frac{\gamma_f - 1}{\gamma_f}} + \left[1 - \left(\frac{P_\infty}{P_b}\right)^{\frac{\gamma_f - 1}{\gamma_f}}\right]\left[\frac{2}{(\gamma_f - 1)M_f^2} + 1\right]\right]^{\frac{1}{2}} dA_f - m_i V_\infty. \qquad (23b)$$

APPENDIX III

SOME REMARKS ON REAL JET ENGINES AND THE USE OF STATIONS " e " AND " f "

In general, the discharge coefficients of engine exhaust nozzles are less than unity. The engine manufacturers assume that the engine behaves as if the flow leaving the final nozzle converges and passes through some area (known as the effective area and denoted by A_f in the present Report) in which the streamlines are all parallel to the undisturbed stream. This area is calculated from a knowledge of the thrust, mass-flow and exhaust temperature, measured on a test-bed, or better still in an altitude test-cell. For estimating flight performance, either the effective area is assumed to be independent of altitude, or an assumed variation is used, or the results obtained from an altitude test-cell are employed. In the present context this variation and the method of obtaining it are immaterial; it is only necessary to allow for the existence of an effective area when constructing the expressions for the various thrusts, whether this area remains constant or not.

In the case of a real nozzle from which the flow contracts and passes through an " effective area," the contracting jet (which is assumed not to mix with the surrounding flow) can support a pressure from the external flow and thus contribute to the total thrust. The problem is unaltered if the boundary of the jet is replaced by a solid surface, so that for the purpose of defining the thrust an idealised picture of the jet may be used, in which the new exit area, at which the flow is parallel to the free-stream, is the effective exit area A_f. As was said earlier, the exit area given by the engine maker is this effective area A_f; expressions for the various thrusts in terms of conditions at station " f " are therefore required.

The important case of flight measurements of thrust must now be considered. These measurements are made in the jet, near the engine outlet, and a method is required for calculating the thrust from them. Now it may be readily shown that it is impossible to separate the internal thrust from the external drag unless the wake surveys are carried out in the plane of the final nozzle (station " e "), and a complete picture obtained of the flow at this position. It is therefore of great importance to have alternative expressions for the thrust in terms of quantities which can be determined in the actual exit plane.

One method that can be adopted is to use the results of the measurements at station " e " to calculate the conditions at station " f," and this method will be used where a knowledge of the effective area is required. But in the case that is probably of the greatest importance, a choked jet exhausting into a region whose pressure is equal to the undisturbed pressure, a simpler approach can be used.* In the absence of any mixing with the external flow the pressure and velocity must be constant along the bounding stream tube of the jet, the pressure being equal to P_∞. Between stations " e " and " f," the following relation holds

$$\int (\rho_e V_e^2 \cos^2 \phi_e + P_e - P_\infty)\, dA_e - \int (\rho_f V_f^2 + P_f - P_\infty)\, dA_f = 0.$$

The difference between the two integrals given above is equal to the resolved component of the force due to the pressure difference $(P - P_\infty)$ acting on the boundary of the jet between the two stations. This difference must be zero, since on the boundary stream tube P is equal to P_∞. Thus the thrust, given in this case by the standard thrust, is obtainable from the first integral in the equation entirely in terms of quantities in the exit plane, where measurements of static and total pressure, temperature *and* angle of flow are required.

If the external pressure is not equal to the free-stream value, the relation given above must be modified. Assuming that the external pressure has the constant value P_b between stations " e " and " f,"

$$\int (\rho_e V_e^2 \cos^2 \phi_e + P_e - P_b)\, dA_e = \int (\rho_f V_f^2 + P_f - P_b)\, dA_f,$$

so that the velocity V_g in the intermediate plane " g " can be related to conditions at station " e."

Thus, for the purposes of the definition and calculation of thrust, when conditions at station " e " are known, it is often not necessary to calculate the conditions at station " f." The choice between stations " e " and " f " is mainly one of convenience. In general, in the absence of measurements, it is almost impossible to estimate conditions at the exit plane, but it is relatively easy for the engine manufacturer to estimate the conditions at station " f."

In many cases, too, the velocity and pressure can be assumed to be uniform across station " f " when estimating thrust from data supplied by the manufacturer, and there is then no need to integrate over the jet area. To *define* the thrusts in terms of uniform flow at station " f " would be unnecessarily restrictive, however, since the exact correspondence between points in stations " e " and " f " connected by a common stream tube would be lost.

*The external pressure surrounding a jet exit under static conditions is not quite equal to the pressure of the undisturbed stream. This is because the jet induces an external flow around the nacelle which changes the external local pressure, particularly at the jet exit. The thrust obtained from engine brochures is ultimately based on test-bed measurements obtained under such conditions, and thus includes any post-exit thrust due to P_b not being equal to P_∞. The importance of this effect is not known.

APPENDIX IV

SOME CAUTIONARY REMARKS REGARDING CENTRAL BODIES AND SIDE INTAKES

All the thrusts (or internal drags) defined in Section 4 are based on the rate of change of momentum of the fluid that flows through the body between a station at infinity upstream and some downstream station. If, therefore, the ducted body has a central body protruding ahead of or behind the main body (*see* Fig. 3), the drag of the portion of the central body protruding ahead of the entry is automatically included in the internal thrust or drag, just as are portions entirely within the main body. The drag of any portion of the central body protruding *behind* the exit is not necessarily included, however; the method to be used in allowing for this drag depends on the definition of thrust that is adopted.

When the central body protrudes a long way ahead of the entry, it is often more convenient to treat the central body as the main body, and to treat the ducted part as a " side intake." This is particularly necessary when the intake does not entirely surround the central body. In such cases the internal drag of the duct, under both definitions, will include the drag of the portions of the main body ahead of the entry that are wetted by the fluid

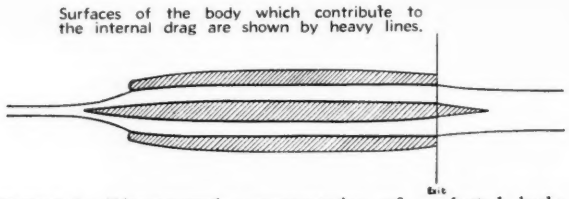


FIGURE 3. Diagrammatic representation of a ducted body having a central body within it.

that flows through the duct. However, it is usual to include the drag of the portions of the main body wetted by the internal flow in the external drag, and to correct for this by reducing the pre-entry drag of the intake by the same amount, to yield, say, a " modified pre-entry drag." That such a procedure is rational is apparent when the ducted parts of the body become so detached from the main body that they can be treated as separate nacelles.

APPENDIX V

FORMULAE FOR INTERNAL DRAG

One of the earliest attempts to define the internal drag of a ducted body was made by Meredith,[6] but he considered only the case when the exit static pressure was equal to the undisturbed pressure. In this case all the definitions compared in Section 5 are equivalent to one another and to Meredith's definition.

Other definitions (unpublished, 1936) covered the case in which the exit static pressure was different from the undisturbed pressure and were equivalent to equation (1) of this report, but in order to apply the definition the velocity V_w was related to the exit velocity V_e and the exit static pressure P_e by the assumption of no transfer of momentum or energy in the wake. This method is now commonly called Jones's method, because Jones made the same assumptions in obtaining the profile drag of an unducted body from measurements in the wake.[5] Taylor[7] has pointed out, however, that the effect of turbulent mixing is not always negligible, particularly when the exit static pressure is appreciably different from the undisturbed pressure. Jones's formula, for constant mass flow through the duct, neglecting compressibility, is

$$D_J = \int \rho V_e \left[V_\infty - V_e \sqrt{\left(1 + \frac{P_e - P_\infty}{\frac{1}{2}\rho V_e^2}\right)} \right] dA_e. \quad (24)$$

After it had been observed that the definition of internal drag given by equation (1) could be expressed in terms of measurable quantities by the use of what was, in effect, Jones's formula, it was natural that Betz's formula[8] should also have been considered, for it had been shown that the two formulae were equivalent to the first order when used to estimate the profile drag of unducted bodies.[9] Betz's formula, which applies to incompressible flow, is

$$D_B = \int \rho V_e \left[V_\infty - V_e \left\{ 1 + 2\frac{P_e - P_\infty}{\rho V_e^2} - \right.\right.$$
$$\left.\left. - \frac{V_\infty^2}{V_e^2}\left(1 - \left(1 - \frac{P_e - P_\infty}{\frac{1}{2}\rho V_\infty^2}\right)^{\frac{1}{2}}\right)\right\}\right] dA_e, \quad (25)$$

and on expanding the radical in powers of $(P_e - P_\infty)/\frac{1}{2}\rho V^2$ this becomes

$$D_B = \int \rho V_e \left[V_\infty - V_e \left\{ 1 + \frac{P_e - P_\infty}{\rho V_e^2} - \right.\right.$$
$$\left.\left. - \frac{1}{2}\left(\frac{P_e - P_\infty}{\rho V_e V_\infty}\right)^2 + \ldots \right\}\right] dA_e. \quad (26)$$

By expanding the radical in equation (24) in powers of $(P_e - P_\infty)/\frac{1}{2}\rho V_e^2$, the equivalent form of Jones's formula is found to be

$$D_J = \int \rho V_e \left[V_\infty - V_e \left\{ 1 + \frac{P_e - P_\infty}{\rho V_e^2} - \right.\right.$$
$$\left.\left. - \frac{1}{2}\left(\frac{P_e - P_\infty}{\rho V_e^2}\right)^2 + \ldots \right\}\right] dA_e. \quad (27)$$

Thus equations (26) and (27) differ only in the terms of order $\left(\frac{P_e - P_\infty}{\rho V_e^2}\right)$ and higher.

Betz's formula has the apparent advantage over Jones's that it gives a real answer when the total pressure at exit is less than the undisturbed static pressure, for which case Jones's formula gives a complex answer. Betz's formula has, in fact, been introduced into internal drag problems (*e.g.* in Ref. 10) in order to cope with this " weakness " of Jones's formula, although Squire in 1936 pointed out that Betz's formula should not be used when Jones's would give a complex answer. Betz's formula was in fact developed for determining the profile drag of closed (unducted) bodies, and has never been extended to give a formula for the internal drag of a ducted body.

It is recommended therefore that Jones's formula should be used if an assessment of the internal drag of a ducted body is required, and that Betz's formula should not be used for this purpose.

REFERENCES

1. ANDREWS, D. R. (1952). Some Considerations of Jet Thrust Measurements in Flight, with particular reference to the Trunnion Thrust Method. R.A.E. Tech. Memo. Aero. 292. A.R.C. 15,265-Perf. 1003-G.T.135-Def. 12. August 1952.

2. STEPHENSON, J., SHIELDS, R. T. and BOTTLE, D. W. (1952). An Investigation into the Pitot Rake Method of Measuring Turbo-Jet Engine Thrust in Flight. Report No. A.A.E.E./Res/265. A.R.C. 15,509-Perf. 1034-G.T. 149-Def. 18, 23rd December 1952.

3. JAKOBSSON, B. (1951). Definition and Measurement of Jet Engine Thrust. *Journal of the Royal Aeronautical Society*, Vol. 55, No. 484, pp. 226-243, April 1951.

4. British Standard Glossary of Aeronautical Terms. B.S. 185: Part I: 1950.

5. The Cambridge University Aeronautics Laboratory (1936). The Measurement of Profile Drag by the Pitot-Traverse Method. R. & M. 1688. January 1936.

6. MEREDITH, F. W. (1935). Note on the Cooling of Aircraft Engines with special reference to Ethylene Glycol Radiators enclosed in Ducts. R. & M. 1683, January 1935.

7. TAYLOR, G. I. (1937). The Determination of Drag by the Pitot Traverse Method. R. & M. 1808, 3rd November 1937.

8. BETZ, A. (1925). Ein Verfahren zur direkten Ermittlung des Profilwidestandes. *Zeitschr. f. Flugtechn. u. Motorluftschifl* 3 Heft S.42 February 1925.

9. GOLDSTEIN, S. ed. (1938). *Modern Development in Fluid Dynamics*. Vol. I, Chap. VI, Section II, §115. Oxford. 1938.

10. BRADFIELD, F. B., ed. (1947). Notes on Technique employed at R.A.E. in Low-Speed Wind-Tunnel Tests in the period 1939-1945. R. & M. 2556, October 1947.

The Stress Analysis of the Circular Conical Fuselage with Flexible Frames

by

W. J GOODEY, M.A., A.F.R.Ae.S.

SUMMARY: This paper is concerned with the problem of the skin-and-stringer-covered conical fuselage of circular cross section, having flexible frames and being subjected to an arbitrary distribution of loads applied in the planes of these frames or on the end cross sections. The skin thickness and stringer cross section may vary from bay to bay but are assumed to be constant in a bay, *i.e.* between any two consecutive frames. The stringers are assumed to be uniformly spaced round the circumference. The frames are assumed to be of uniform cross section (circumferentially) but their stiffnesses are arbitrary, as is also their spacing along the fuselage.

This paper is a sequel to one previously published in the JOURNAL[1], but the method of analysis is now by minimum strain energy instead of by deflections. Owing to the generality of the investigation it is impossible to present the solution—in terms of the known loads and stiffnesses—in any explicit form, and attention has therefore been concentrated on the technique of numerical computation. A scheme of tabulation has been evolved, and its use is illustrated by a fully-worked numerical example of a moderately complex type.

1. Introduction

It is now well known that the stress distribution in a beam-like structure such as an aeroplane fuselage, where the cross-sectional dimensions are not small compared with the length, is liable to be very different from that given by the elementary theories of bending and torsion. A number of papers have been published on this subject, among them one by the present author[1] bearing the somewhat deceptive title "The Stresses in a Circular Fuselage"*.

In spite of its title it must now be admitted that this paper did little more than demonstrate the existence of the phenomenon, though it may have served a useful purpose if it did only this.

The present investigation is of a much more general character, and is concerned with the problem of the skin-and-stringer-covered conical fuselage of circular cross section, having flexible frames and being subjected to an arbitrary distribution of loads applied in the planes of these frames or on the end cross sections. A typical problem of this type is illustrated diagrammatically in Fig. 1.

The basic assumptions governing the analysis are as follows.

(*i*) The surface of the fuselage forms part of a right circular cone, the frames being in planes normal to the axis of the cone.

(*ii*) The cross section of a frame is uniform, but the cross-sectional dimensions, and therefore the stiffness, may vary from frame to frame. The stiffness which is of primary concern is the resistance of a frame to distortion in its own plane; any resistance to bending or twisting out of the plane will be neglected. The spacing of the frames along the axis of the fuselage is arbitrary.

(*iii*) A part of the fuselage consisting of a frame and the structure between this frame and the next one (away from the vertex of the cone) will be referred to as a "bay." The covering of the fuselage in a bay consists of a skin of uniform thickness, reinforced with stringers of uniform cross section, uniformly spaced round the circumference. Thus, the stringers lie along generating lines of the cone. The skin thickness and stringer cross section may vary from bay to bay.

(*iv*) The stringers will be assumed to be sufficiently closely spaced to be regarded as a continuously distributed reinforcing of uniform thickness. Bending and torsional stiffness of the individual stringers (in association with a piece of skin) will be neglected.

(*v*) As already mentioned, the applied loads are either in the planes of the frames, or on the end cross sections, but are otherwise arbitrary. The analysis could in fact be made to cover practically any type of loading, but it is convenient to impose this restriction, which involves only a slight loss in generality.

Any further assumptions of a secondary nature which it may be desirable to make will be brought in as the occasion arises.

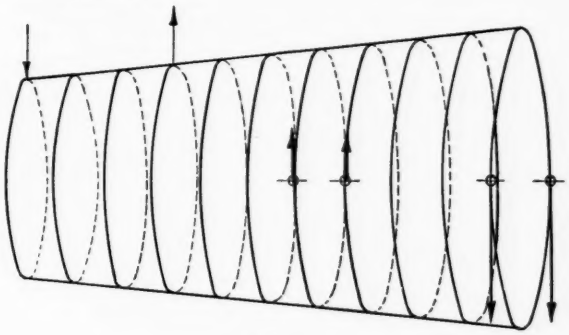


FIGURE 1.

Originally received November 1953.

*See also Ref. 2 which includes an interesting review of current literature on the subject.

Notation

A_j total cross section of end-load carrying material

A in Section 10, symmetrical matrix of structural coefficients

C modulus of rigidity

d_j depth between the centroids of the two flanges of frame j

d_{wj} effective depth of web of frame j

E Young's modulus

$\overline{G}_{nj}$ defined in equation (21)

$\overline{H}_{nj}$ defined in equation (21)

h_j offset of the neutral axis of the cross section of frame j from the skin line

I_j moment of inertia of cross section of frame j, for bending in its own plane

$\overline{J}_{nj}$ defined in equation (20)

j bay j consisting of frame j and skin and stringers between frames j and $j+1$

$\overline{K}_{nj}$ defined in equation (20)

L_j length of bay $j=z_{j+1}-z_j$

L in Section 10, column of loading terms

l_{nj} term in strain energy equation nj due to the applied loading

M bending moment in a frame, at the cross section defined by angle θ (*see* Fig. 3)

$\overline{M}$ value of M when applied loads are balanced by skin loading of elementary bending and torsion theory

$\overline{M}_{Fj}$ externally applied moment loading, positive anti-clockwise looking along Oz away from O, on frame j (*see* Fig. 3)

m_0, m_1, m_2 arbitrary constants defined in equation (29)

N direct stress loading (*i.e.* tension per unit width) along the generating lines, taken by the skin and stringers

N_{Fj} externally applied radial loading, positive outwards, on frame j

P_{nj} defined in equation (18)

$\overline{P}_{nj}$ defined in equation (19)

Q_{nj} defined in equation (18)

$\overline{Q}_{nj}$ defined in equation (19)

q shear loading (shear per unit length) in the skin at section z, parallel and normal to the generating lines, positive as in Fig. 2

q_{Fj} shear loading applied to frame j by the skin, positive when acting in the direction of increasing θ, *i.e.* anti-clockwise looking along z-axis away from origin (*see* Fig. 3)

$\overline{q}_{Fj}$ externally applied tangential loading (at skin radius R_j) on frame j

R radius to the skin surface at section z

R_j value of R at frame j

S shear in a frame at the cross section defined by angle θ (*see* Fig. 3)

$\overline{S}$ value of S corresponding to $\overline{M}$

T end load in a frame at the cross section defined by angle θ (*see* Fig. 3)

$\overline{T}$ value of T corresponding to $\overline{M}$

T, T' in Section 10, lower triangular matrix and its transpose

t_j skin thickness in bay j

t_{Nj} effective thickness of end load carrying material in bay j

t_{wj} web thickness of frame j

U strain energy of complete structure

U_j strain energy of bay j

U_{Fj} total strain energy of frame j

U_{MFj} strain energy of bending of frame j

U_{Nj} strain energy due to loading N in bay j

U_{SFj} shear strain energy in frame j

U_{Sj} shear strain energy in the skin of bay j

U_{TFj} strain energy of end load in frame j

v tangential deflection of the point on the neutral axis of the frame defined by angle θ, taken positive in direction of increasing θ

W load applied to a frame, in Section (8)

w radial deflection of the point on the neutral axis of the frame defined by angle θ, taken positive outwards

x distance of section z from the frame just behind it (*i.e.* looking towards O). For bay j, $x=z-z_j$ $^{(0\leqslant x\leqslant L_j)}$

Y column of subsidiary variables

z distance of any cross section normal to the axis of the cone from its apex O

α semi-angle of the cone $=\tan^{-1} R/z$

η_j ratio of total flange area of frame j to effective web area $=I_j d_{wj}/\rho_j^2 t_{wj} d_j^2$

θ angle between plane zOA and a fixed plane through zO (positive anti-clockwise, measuring from the "downward vertical," and looking in the direction of z increasing) (*see* Fig. 3)

$\lambda_j=(R_{j+1}-R_j)/(R_{j+1}+R_j)$

ρ_j radius of gyration of the cross section of frame j

τ_j torque about the z-axis, in bay j

$\left.\begin{matrix}\phi_{nj}\\ \psi_{nj}\end{matrix}\right\}$ defined in equation (7)

2. The Equations of Equilibrium

The position of any point A on the skin surface may be specified by the co-ordinate z and the angle θ between the plane zOA and a fixed plane through zO. This fixed plane can be conveniently regarded as the "vertical" plane.

Referring to Fig. 2, consider now the equilibrium of the element $ABCD$ of the shell bounded by sections z and $z+\delta z$ and the generating lines defined by the angles θ and $\theta+\delta\theta$.

Resolving along the normal to the element there is no circumferential stress in the skin, since q and N are both in the tangent plane. Bending stiffness of the stringers (+skin), which would make possible the existence of shear forces normal to $ABCD$, is neglected, while normal pressure on the skin is assumed to be zero.

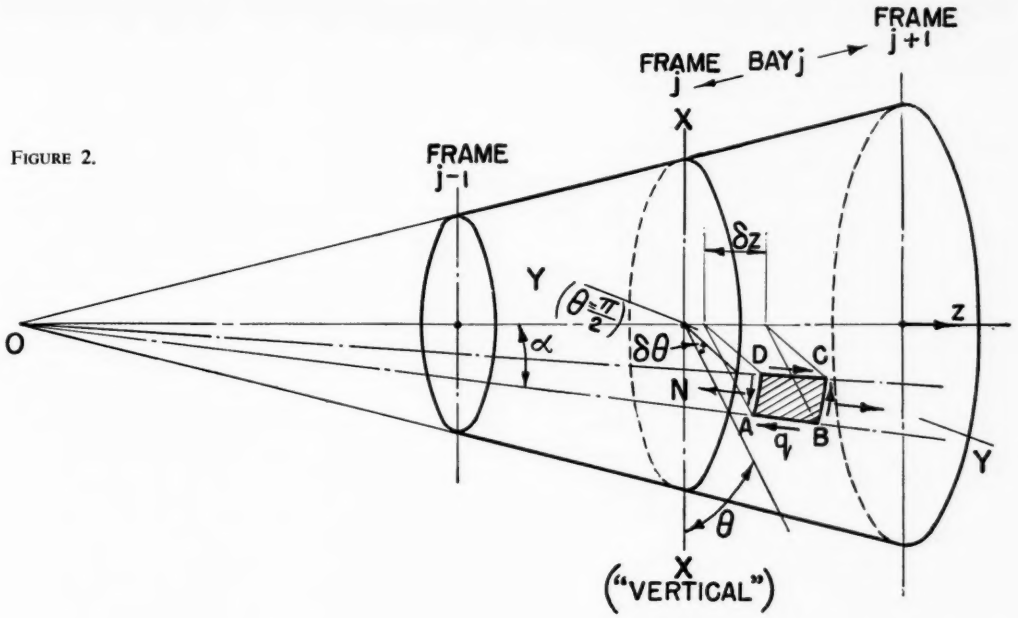
Resolving along the generating line AB,

$$\frac{\partial}{\partial z}(NR)+\frac{\partial}{\partial\theta}(q\sec\alpha)=0 \quad . \quad . \quad (1)$$

and taking moments about O,

$$\frac{\partial}{\partial z}(qRz)=0 \quad . \quad . \quad . \quad (2)$$

FIGURE 2.



(The same equation comes from resolving along AD, but the derivation is less elegant.)

Thus, qRz is independent of z or, since z/R is constant,

$$qR^2 \text{ is independent of } z \quad . \quad . \quad (3)$$

This only applies between one frame and the next, since there may be tangential loads applied to the skin at a frame, so equation (2) is not valid if the element $ABCD$ includes a portion of a frame.

Hence qR^2 is a function of the angle θ and of the bay number j.

Equation (1) may now be written in the form

$$\frac{\partial}{\partial z}(NR\cos\alpha) = -\frac{1}{R^2}\frac{\partial}{\partial\theta}(qR^2),$$

which can then be integrated with respect to z. However, to make the formulae applicable to the limiting case of a circular cylinder (when the point O moves away to infinity and R becomes constant) it is convenient to replace ∂z in this last equation by ∂x. It is then integrated with respect to x, adjusting the "constant" of integration (a function of θ) to make $N=N_j$ at frame j ($x=0$). The result is that

$$NR\cos\alpha = N_jR_j\cos\alpha - \frac{\partial}{\partial\theta}(qR^2)\int_0^x \frac{dx}{R^2}. \quad (4)$$

now $$\int_0^x \frac{dx}{R^2} = \frac{dx}{dR}\int_{R_j}^{R}\frac{dR}{R^2} = \frac{dx}{dR}\left(\frac{1}{R_j}-\frac{1}{R}\right) = \frac{x}{RR_j},$$

since $$\frac{dx}{dR} = \frac{x}{R-R_j}.$$

Therefore

$$NR\cos\alpha = N_jR_j\cos\alpha - \frac{x}{RR_j}\frac{\partial}{\partial\theta}(qR^2) \quad (5)$$

(*Note.* When $\alpha=0$, $R=$constant, so that N then varies linearly with x between frames.)

When $x=L_j$, $N=N_{j+1}$, and so from (5)

$$\frac{\partial}{\partial\theta}(qR^2) = \frac{R_jR_{j+1}}{L_j}(N_jR_j\cos\alpha - N_{j+1}R_{j+1}\cos\alpha) \quad . \quad . \quad . \quad (6)$$

Given N_j and N_{j+1} this equation enables qR^2 (for bay j) to be calculated, by integration with respect to θ, the constant of integration being adjusted to give the correct torque about Oz.

3. Types of Stress Distribution

Let $$N_jR_j\cos\alpha = \sum_{n=0}\psi_{nj}\cos n\theta + \sum_{n=1}\phi_{nj}\sin n\theta \quad . \quad (7)$$

θ is taken to lie in the range $-\pi$ to $+\pi$. The upper limit of n is omitted from the summation signs because, although in theory it will generally be infinite, in practice only enough terms are retained to give the required degree of accuracy. The bending moment about the axis XX (*see* Fig. 2), at frame j, is equal to

$$\int_{-\pi}^{+\pi}(N_jR_j\cos\alpha)\,R_j\sin\theta\,d\theta = \pi R_j\phi_{1,j},$$

positive moment giving tension on the side $\theta=0$ to π. Similarly, the bending moment about YY

$$= \int_{-\pi}^{\pi}(N_jR_j\cos\alpha)\,R_j\cos\theta\,d\theta = \pi R_j\psi_{1,j},$$

positive moment giving tension in the range

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

and the end load (in the z-direction)

$$= \int_{-\pi}^{\pi} (N_j R_j \cos\alpha)\, d\theta = 2\pi\psi_{0,j} \text{ (tension positive).}$$

The cosine terms (including $\psi_{0,j}$) in equation (7) represent a loading (N_j) which is symmetrical with respect to the plane XX, the terms with odd multiples of θ being anti-symmetrical with respect to the plane YY (since $\sin n(\theta - \pi/2) = \pm\cos n\theta$ when n is odd).

The sine terms give a loading N_j which is anti-symmetrical with respect to the plane XX, the terms with odd multiples of θ being symmetrical with respect to the plane YY (since $\cos n(\theta - \pi/2) = \pm\sin n\theta$ when n is odd). The terms $(\psi_{0,j} + \psi_{1,j}\cos\theta + \phi_{1,j}\sin\theta)$ in equation (7) clearly represent stress distributions of the "elementary bending theory" type, the first being a uniformly distributed end load, the second a bending moment about YY, and the third a bending moment about XX. All the other terms are self-equilibrating distributions, and can be regarded as corrections to the elementary theory. These self-equilibrating distributions are essentially of the nature of redundancies, and the coefficients $\psi_{2,j} \ldots \psi_{nj} \ldots$; $\phi_{2,j} \ldots \phi_{nj} \ldots$, which determine the amount of each distribution present, may be determined by the principle of minimum strain energy. In practice, calculations will be limited to finding only the first few terms in the series, which is usually fairly rapidly convergent.

But first a formula must be derived for the skin shear loading in terms of the ψ's and ϕ's, using equations (6) and (7). From these,

$$\frac{\partial}{\partial\theta}(qR^2) = \frac{R_j R_{j+1}}{L_j}\left\{\sum_{n=0}(\psi_{nj} - \psi_{n,j+1})\cos n\theta + \sum_{n=1}(\phi_{nj} - \phi_{n,j+1})\sin n\theta\right\},$$

whence, on integration with respect to θ,

$$qR^2 = \frac{R_j R_{j+1}}{L_j}\left\{\sum_{n=1}(\psi_{nj} - \psi_{n,j+1})\frac{\sin n\theta}{n} - \sum_{n=1}(\phi_{nj} - \phi_{n,j+1})\frac{\cos n\theta}{n}\right\} + \frac{\tau_j}{2\pi}, \qquad (8)$$

where τ_j is independent of θ, being equal to $\int_{-\pi}^{\pi} qR^2 d\theta$, and is therefore the torque about zO.

Note that the coefficient $(\psi_{0,j} - \psi_{0,j+1})$ is omitted from equation (8). This coefficient would give a term

$$qR^2 = \frac{R_j R_{j+1}}{L_j}(\psi_{0,j} - \psi_{0,j+1})\theta,$$

which would make q discontinuous at $\theta = \pm\pi$. Now by implication the end load at frame j is continuous (in the z-direction), since otherwise the quantity $\psi_{0,j}$ would be two-valued, $\psi_{0,j}+$, say, on one side of the frame and $\psi_{0,j}-$ on the other side. This possibility is excluded by assumption (v) of Section 1. Thus, either $(\psi_{0,j} - \psi_{0,j+1}) = 0$, or there must have been a change of end load *between* frames j and $j+1$. This change could only be brought about by the application of load along a finite number of generating lines of the surface, since the equations of equilibrium imply that there is no distributed loading of the skin-friction type. But loading along generating lines has also been excluded by assumption (v) of Section 1, and hence $\psi_{0,j} = \psi_{0,j+1}$.

Equations (7) and (8) give the most general form of stress distribution in the skin and stringers on the assumptions made in Section 1, and the strain energy of the various parts of the structure may now be calculated. For this purpose, and for the calculation of the strain energy derivatives with respect to the unknowns ψ_{nj} and ϕ_{nj} ($n = 2, 3$, etc.), it has been found very convenient to use matrix notation. Only the most elementary parts of the theory of matrices are required, up to the differentiation of quadratic forms.

4. The Strain Energy of Shear in the Skin

(It is assumed that the skin, stringers and frames are made of the same material throughout. If not, the appropriate modifications to the formulae must be made, though probably the simplest way of dealing with this is to use equivalent thicknesses and cross-sectional areas all in the same material.)

Then $$6EU_{sj} = 6E\int_{x=0}^{L_j}\int_{-\pi}^{\pi}\left(\frac{q}{t_j}\right)^2 \frac{t_j R\sec\alpha\, d\theta\, dx}{2C}$$

$$= \frac{3E\sec\alpha}{Ct_j}\int_0^{L_j}\int_{-\pi}^{\pi}\frac{(qR^2)^2\, d\theta\, dx}{R^3}.$$

The factor $6E$ is put in for convenience in numerical work.

Now

$$\int_0^{L_j}\frac{dx}{R^3} = \cot\alpha\int_{R_j}^{R_{j+1}}\frac{dR}{R^3} = \frac{\cot\alpha(R_{j+1}^2 - R_j^2)}{2R_j^2 R_{j+1}^2} = \frac{L_j(R_j + R_{j+1})}{2R_j^2 R_{j+1}^2}$$

Therefore

$$6EU_{sj} = \frac{3EL_j(R_j + R_{j+1})\sec\alpha}{2Ct_j R_j^2 R_{j+1}^2}\int_{-\pi}^{\pi}(qR^2)^2\, d\theta,$$

which using (8)

$$= \frac{3E\pi(R_j + R_{j+1})\sec\alpha}{2Ct_j L_j}\left[\sum_1\frac{1}{n^2}(\psi_{nj} - \psi_{n,j+1})^2 + \sum_1\frac{1}{n^2}(\phi_{nj} - \phi_{n,j+1})^2\right] + \text{a multiple of } \tau_j^2.$$

The $n = 1$ terms and τ_j are known from the applied loads, and so they will disappear on differentiation with respect to any of the unknowns. Thus, only the unknown, or variable, part of $6EU_{sj}$ need be considered. Moreover, expressions need only be obtained for the partial differential coefficients of $6EU_{sj}$ with respect to the ψ's, since those with respect to the ϕ's will be obtainable therefrom, simply by replacing each ψ by the corresponding ϕ.

Now

$$\psi_{nj}-\psi_{n,j+1}=[1,-1]\begin{bmatrix}\psi_{nj}\\ \psi_{n,j+1}\end{bmatrix}, \text{ and hence}$$

$$6EU_{Sj}=\frac{3E\pi(R_j+R_{j+1})\sec\alpha}{2Ct_jL_j}\times$$
$$\times\sum_{n=1}\frac{1}{n^2}\left[\psi_{nj},\psi_{n,j+1}\right]\begin{bmatrix}1\\ -1\end{bmatrix}\left[1,-1\right]\begin{bmatrix}\psi_{nj}\\ \psi_{n,j+1}\end{bmatrix}$$

+a similar expression with ϕ's instead of ψ's
+a multiple of τ_j^2.

Differentiating with respect to ψ_{nj} and $\psi_{n,j+1}$ in turn, and denoting the pair of differential coefficients—for a particular value of n—by $6E\partial U_{Sj}/\partial\psi_n$,

$$6E\frac{\partial U_{Sj}}{\partial\psi_n}=\frac{3E\pi(R_j+R_{j+1})\sec\alpha}{Ct_jL_jn^2}\begin{bmatrix}1\\ -1\end{bmatrix}\left[1,-1\right]\begin{bmatrix}\psi_{nj}\\ \psi_{n,j+1}\end{bmatrix}$$
$$=\frac{3E\pi(R_j+R_{j+1})\sec\alpha}{Ct_jL_j}\begin{bmatrix}\frac{1}{n^2},-\frac{1}{n^2}\\ -\frac{1}{n^2},\frac{1}{n^2}\end{bmatrix}\begin{bmatrix}\psi_{nj}\\ \psi_{n,j+1}\end{bmatrix}\quad(9)$$

$(n=2, 3, \text{and so on})$

These differential coefficients could have been written down easily enough without using matrix notation, since the expression for the strain energy in the skin is a particularly simple one. The advantage of using this notation will, however, become more evident in the more complicated expressions for the strain energy of the stringers and the frames. Note that the top row of (9) is the differential coefficient with respect to ψ_{nj}, while the bottom row is the differential coefficient with respect to $\psi_{n,j+1}$. Note also that the form of (9) is independent of the applied loading, and for this reason the strain energy terms—*i.e.* the *coefficients* of the ψ's—will be called "structural terms." Terms whose form depends on the applied loading will be called "loading terms."

5. The Strain Energy of the Stringers

Let U_{Nj} be the strain energy due to the loading N, in bay j. This loading is along the generating lines of the cylindrical shell, and is carried as end load in the stringers. Each stringer may usually be assumed to have a piece of the skin, to which it is attached, acting with it as an end-load carrying member, thus giving an effective stringer area somewhat greater than the area of the stringer alone. Assuming these effective stringer areas to be spread uniformly round the circumference, let t_{Nj} be the effective thickness of end-load carrying material so distributed.

Since, in practice, the number of stringers and the effective cross-sectional area of each stringer will be fixed quantities in a bay, the value of t_{Nj} will depend on R. Thus, if A_j denote the total cross section of end-load carrying material in bay j, then

$$t_{Nj}=\frac{A_j}{2\pi R}. \quad . \quad . \quad . \quad (10)$$

and
$$6EU_{Nj}=6E\int_{x=0}^{L_j}\int_{-\pi}^{\pi}\left(\frac{N}{t_{Nj}}\right)^2\frac{t_{Nj}R\sec\alpha\,d\theta\,dx}{2E}$$
$$=\frac{6\pi\sec^3\alpha}{A_j}\int_0^{L_j}\int_{-\pi}^{\pi}(NR\cos\alpha)^2\,d\theta\,dx.$$

Now from equations (5), (7) and (8)

$$NR\cos\alpha=\left(1-\frac{x}{L_j}\right)\frac{R_j}{R}\sum_{n=0}(\psi_{nj}\cos n\theta+\phi_{nj}\sin n\theta)+$$
$$+\frac{x}{L_j}\frac{R_{j+1}}{R}\sum_{n=0}(\psi_{n,j+1}\cos n\theta+\phi_{n,j+1}\sin n\theta)\quad . \quad (11)$$

Hence $6EU_{Nj}=$
$$\frac{6\pi^2\sec^3\alpha}{A_j}[a_j\sum_{n=1}(\psi_{nj}^2+\phi_{nj}^2)+2b_j\sum_{n=1}(\psi_{nj}\psi_{n,j+1}+\phi_{nj}\phi_{n,j+1})+$$
$$+c_j\sum_{n=1}(\psi_{n,j+1}^2+\phi_{n,j+1}^2)+\psi_{0,j}\psi_{0,j+1}\text{ terms}]$$
$$. \quad . \quad . \quad (12)$$

where

$$a_j=\int_0^{L_j}\left(1-\frac{x}{L_j}\right)^2\frac{R_j^2}{R^2}dx,\quad b_j=\int_0^{L_j}\frac{x}{L_j}\left(1-\frac{x}{L_j}\right)\frac{R_jR_{j+1}}{R^2}dx,$$

and

$$c_j=\int_0^{L_j}\frac{x^2}{L_j^2}\frac{R_{j+1}^2}{R^2}dx.$$

The values of these integrals may be found by expressing the integrands in terms of R, and integrating between the limits R_j and R_{j+1}. Omitting the intermediate steps and merely noting that

$$dx=L_jdR/(R_{j+1}-R_j)$$

and
$$x/L_j=(R-R_j)/(R_{j+1}-R_j),$$

the following results are obtained:

$$\frac{a_j}{L_j}=\frac{R_j^2}{(R_{j+1}-R_j)^2}\left(\frac{R_{j+1}+R_j}{R_j}-\frac{2R_{j+1}}{R_{j+1}-R_j}\log_e\frac{R_{j+1}}{R_j}\right),$$

$$\frac{b_j}{L_j}=\frac{R_jR_{j+1}}{(R_{j+1}-R_j)^2}\left(\frac{R_{j+1}+R_j}{R_{j+1}-R_j}\log_e\frac{R_{j+1}}{R_j}-2\right),$$

and

$$\frac{c_j}{L_j}=\frac{R_{j+1}^2}{(R_{j+1}-R_j)^2}\left(\frac{R_{j+1}+R_j}{R_{j+1}}-\frac{2R_j}{R_{j+1}-R_j}\log_e\frac{R_{j+1}}{R_j}\right).$$

These expressions are not very convenient for practical use, particularly if $(R_{j+1}-R_j)$ is small compared with R_j, since they take the indeterminate form $0/0$ when $R_{j+1}=R_j$. In order to avoid this difficulty, and to obtain approximate values for a_j, b_j and c_j which may be used when—as will usually be the case—R_{j+1} differs only slightly from R_j, we write

$$\frac{R_{j+1}-R_j}{R_{j+1}+R_j}=\lambda_j. \quad . \quad . \quad . \quad (13)$$

The logarithmic term may then be expanded as a power series in λ_j, and it is found that

$$\frac{a_j}{L_j} = (1-\lambda_j)\left(\frac{1}{3} + \frac{\lambda_j^2}{3\cdot 5} + \frac{\lambda_j^4}{5\cdot 7} + \ldots.\right),$$

$$\frac{b_j}{L_j} = \frac{1}{6} - \left(\frac{\lambda_j^2}{3\cdot 5} + \frac{\lambda_j^4}{5\cdot 7} + \ldots.\right),$$

and

$$\frac{c_j}{L_j} = (1+\lambda_j)\left(\frac{1}{3} + \frac{\lambda_j^2}{3\cdot 5} + \frac{\lambda_j^4}{5\cdot 7} + \ldots.\right).$$

In practice the square and higher powers of λ_j may be ignored, and it will be sufficient to take

$$\left.\begin{aligned} \frac{a_j}{L_j} &= \frac{1-\lambda_j}{3} = \frac{2R_j}{3(R_j+R_{j+1})}, \\ \frac{b_j}{L_j} &= \frac{1}{6} \\ \text{and} \quad \frac{c_j}{L_j} &= \frac{1+\lambda_j}{3} = \frac{2R_{j+1}}{3(R_j+R_{j+1})} \end{aligned}\right\} \quad . \quad . \quad (14)$$

Equation (12) then takes the form

$$6EU_{Nj} = \frac{2\pi^2 L_j \sec^3\alpha}{A_j} \sum_{n=1} \{(1-\lambda_j)\psi_{nj}^2 + \psi_{nj}\psi_{n,j+1} + (1+\lambda_j)\psi_{n,j+1}^2\}$$

$$+ \text{a similar expression with } \phi\text{'s instead of } \psi\text{'s}$$

$$+ \psi_{0,j},\ \psi_{0,j+1} \text{ terms.} \quad . \quad . \quad . \quad (15)$$

The variable part of $6EU_{Nj}$ may be expressed in matrix notation as

$$\frac{\pi^2 L_j \sec^3\alpha}{A_j} \sum_{n=2} \begin{bmatrix} \psi_{nj}, \psi_{n,j+1} \end{bmatrix} \begin{bmatrix} 2(1-\lambda_j), & 1 \\ 1 & , 2(1+\lambda_j) \end{bmatrix} \begin{bmatrix} \psi_{nj} \\ \psi_{n,j+1} \end{bmatrix}$$

$$+ \text{a similar expression with } \phi\text{'s instead of } \psi\text{'s,}$$

and hence

$$6E\frac{\partial U_{Nj}}{\partial \psi_n} = \frac{2\pi^2 L_j \sec^3\alpha}{A_j} \begin{bmatrix} 2(1-\lambda_j), & 1 \\ 1 & , 2(1+\lambda_j) \end{bmatrix} \begin{bmatrix} \psi_{nj} \\ \psi_{n,j+1} \end{bmatrix} \quad (16)$$

6. Bending Moment, Shear, and End Load in Frame j

It is usual in investigations of this nature to take into account only the strain energy of bending of the frames in their own planes, since it is argued that, unless the depth of the frame cross section is appreciable compared with the radius (R_j), the strain energy of end-load and shear is likely to be small compared with the bending strain energy.

It was originally decided to make this assumption with the refinement that allowance would be made for any offset of the neutral axis of the frame cross section from the skin line, since this may have an appreciable effect on the frame bending moment. Further investigation, however, showed that if the offset of the neutral axis was allowed for, the strain energy of end-load in the frame should be taken into account at the same time, for otherwise the computed stress distribution might be seriously in error. Finally, it was decided for the sake of completeness to include the shear strain energy of the frame.

It is worth noting at this stage that, although frame bending moments will generally be less than would be obtained by the application of the "elementary" theory (*i.e.* assuming a sinusoidal distribution of skin shear), the skin shear—and therefore the end load and shear in a frame—will probably be considerably higher than the "elementary" theory would predict in the neighbourhood of a concentrated applied load. Thus, the bending strain energy will tend to be a lower proportion, and the end load and shear strain energy a higher proportion, of the total strain energy of the frame.

6.1. THE LOADING APPLIED TO THE FRAME

The loading (q_{Fj}) applied to frame j by the skin is given by

$$q_{Fj}R_j^2 = qR^2(\text{bay } j) - qR^2(\text{bay } j-1)$$

Substituting for the qR^2's from equation (8), and dividing throughout by R_j,

$$q_{Fj}R_j = \sum_{n=1} \left\{ -\frac{R_{j-1}}{L_{j-1}}\psi_{n,j-1} + \left(\frac{R_{j-1}}{L_{j-1}} + \frac{R_{j+1}}{L_j}\right)\psi_{nj} - \right.$$

$$\left. - \frac{R_{j+1}}{L_j}\psi_{n,j+1} \right\} \frac{\sin n\theta}{n} -$$

$$- \sum_{n=1} \left\{ -\frac{R_{j-1}}{L_{j-1}}\phi_{n,j-1} + \left(\frac{R_{j-1}}{L_{j-1}} + \frac{R_{j+1}}{L_j}\right)\phi_{nj} - \right.$$

$$\left. - \frac{R_{j+1}}{L_j}\phi_{n,j+1} \right\} \frac{\cos n\theta}{n} +$$

$$+ \frac{\tau_j - \tau_{j-1}}{2\pi R_j} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$$= (\text{say}) \sum_{n=0} (P_{nj}\sin n\theta + Q_{nj}\cos n\theta) \quad . \quad . \quad . \quad (18)$$

In addition to the skin loading there may be external loads applied to the frame. These may be resolved into radial and tangential components, and moments in the plane of the frame, and it will be assumed that each of these types of loading may be expressed in the form of a trigonometrical series of the type in equation (18). As will be seen later, even point loads or moments may be dealt with in this manner. The term "loading" means the amount of load or moment applied per unit length of circumference measured at the skin line.

Let

$$\bar{q}_{Fj}R_j = \sum_{n=0} (\bar{P}_{nj}\sin n\theta + \bar{Q}_{nj}\cos n\theta) \quad . \quad (19)$$

$$\bar{N}_{Fj}R_j = \sum_{n=0} (\bar{J}_{nj}\sin n\theta + \bar{K}_{nj}\cos n\theta) \quad . \quad (20)$$

and

$$\bar{M}_{Fj} = \sum_{n=0} (\bar{G}_{nj}\sin n\theta + \bar{H}_{nj}\cos n\theta) \quad . \quad (21)$$

The total tangential loading applied to the frame is $(q_{Fj} + \bar{q}_{Fj})$.

6.2. THE EQUATIONS OF EQUILIBRIUM OF THE FRAME

The frame (Fig. 3) is assumed to have a finite depth of cross section which is fairly small compared with R_j.

Consider an element of the frame between the radii θ and $\theta + \delta\theta$.

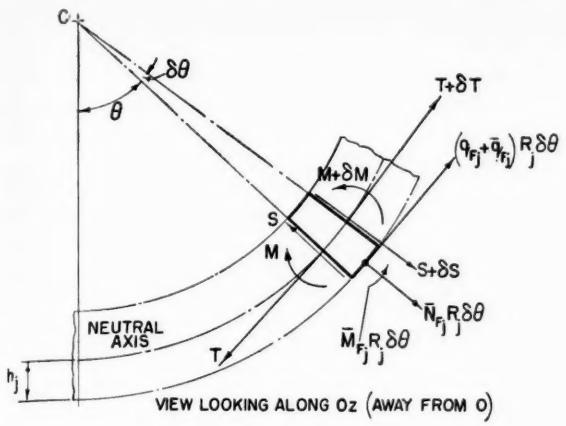


FIGURE 3.

Resolving tangentially, and proceeding to the limit $\delta\theta \to 0$, gives

$$(q_{Fj}+\bar{q}_{Fj})R_j+\frac{dT}{d\theta}+S=0 \quad . \quad . \quad (22)$$

Resolving radially gives

$$\bar{N}_{Fj}R_j+\frac{dS}{d\theta}-T=0 \quad . \quad . \quad (23)$$

and taking moments about C gives

$$\bar{M}_{Fj}R_j+(q_{Fj}+\bar{q}_{Fj})R_j^2+(R_j-h_j)\frac{dT}{d\theta}+\frac{dM}{d\theta}=0 \quad (24)$$

From (22) and (24), eliminating $dT/d\theta$,

$$(R_j-h_j)S=\frac{dM}{d\theta}+(q_{Fj}+\bar{q}_{Fj})R_jh_j+\bar{M}_{Fj}R_j \quad . \quad (25)$$

and then, from (23) and (25)

$$(R_j-h_j)T=\frac{d^2M}{d\theta^2}+R_jh_j\frac{d}{d\theta}(q_{Fj}+\bar{q}_{Fj})+R_j\frac{d\bar{M}_{Fj}}{d\theta}+ +R_j(R_j-h_j)\bar{N}_{Fj} \quad . \quad (26)$$

These last two equations enable S and T to be found when M is known.

Further, from (24) and (26), eliminating T,

$$\frac{d^3M}{d\theta^3}+\frac{dM}{d\theta}+(q_{Fj}+\bar{q}_{Fj})R_j^2+R_jh_j\frac{d^2}{d\theta^2}(q_{Fj}+\bar{q}_{Fj})+ +R_j(R_j-h_j)\frac{d\bar{N}_{Fj}}{d\theta}+\left(\bar{M}_{Fj}+\frac{d^2\bar{M}_{Fj}}{d\theta^2}\right)R_j=0 \quad (27)$$

6.3. THE SOLUTION OF EQUATION (27). THE FRAME BENDING MOMENT

Substituting from equations (18), (19), (20) and (21), the result may be put in the form

$$\frac{d^3M}{d\theta^3}+\frac{dM}{d\theta}+\sum_{n=1}\{(P_{nj}+\bar{P}_{nj}+\bar{G}_{nj})R_j-n(R_j-h_j)\bar{K}_{nj}- -n^2(P_{nj}h_j+\bar{P}_{nj}h_j+\bar{G}_{nj}R_j)\}\sin n\theta + \sum_{n=0}\{(Q_{nj}+\bar{Q}_{nj}+\bar{H}_{nj})R_j+n(R_j-h_j)\bar{J}_{nj}- -n^2(Q_{nj}h_j+\bar{Q}_{nj}h_j+\bar{H}_{nj}R_j)\}\cos n\theta=0 \quad . \quad . \quad (28)$$

By considering the equilibrium of the complete frame it can be shown that the $n=0$ and $n=1$ terms in this last equation are missing.

Thus, resolving the load on the frame in the direction XX (*see* Fig. 2),

$$\int_{-\pi}^{\pi}\{(q_{Fj}+\bar{q}_{Fj})\sin\theta-\bar{N}_{Fj}\cos\theta\}R_j\,d\theta=0,$$

whence $P_{1,j}+\bar{P}_{1,j}-\bar{K}_{1,j}=0$, while the coefficient of $\sin\theta$ in equation (28) is $(P_{1,j}+\bar{P}_{1,j}-\bar{K}_{1,j})(R_j-h_j)$.

Similarly, resolving in the direction YY,

$$\int_{-\pi}^{\pi}\{(q_{Fj}+\bar{q}_{Fj})\cos\theta+\bar{N}_{Fj}\sin\theta\}R_j\,d\theta=0,$$

whence $$Q_{1,j}+\bar{Q}_{1,j}+\bar{J}_{1,j}=0,$$

while the coefficient of $\cos\theta$ in equation (28) is

$$(Q_{1,j}+\bar{Q}_{1,j}+\bar{J}_{1,j})(R_j-h_j).$$

Finally, taking moments about C (*see* Fig. 3) gives

$$\int_{-\pi}^{\pi}\{(q_{Fj}+\bar{q}_{Fj})R_j+\bar{M}_{Fj}\}R_j\,d\theta=0,$$

whence $$Q_{0,j}+\bar{Q}_{0,j}+\bar{H}_{0,j}=0,$$

while the constant term in equation (28) is

$$(Q_{0,j}+\bar{Q}_{0,j}+\bar{H}_{0,j})R_j.$$

The general solution of (28) is therefore

$$M=-\sum_{n=2}\{(P_{nj}+\bar{P}_{nj}+\bar{G}_{nj})R_j-n(R_j-h_j)\bar{K}_{nj}- -n^2(P_{nj}h_j+\bar{P}_{nj}h_j+\bar{G}_{nj}R_j)\}\frac{\cos n\theta}{n(n^2-1)}+ +\sum_{n=2}\{(Q_{nj}+\bar{Q}_{nj}+\bar{H}_{nj})R_j+n(R_j+h_j)\bar{J}_{nj}- -n^2(Q_{nj}h_j+\bar{Q}_{nj}h_j+\bar{H}_{nj}R_j)\}\frac{\sin n\theta}{n(n^2-1)} +(m_0+m_1\sin\theta+m_2\cos\theta) \quad . \quad . \quad . \quad . \quad . \quad (29)$$

The values of m_0, m_1 and m_2 may be found by considering the deflection of the frame. The equation for the radial deflection (w) is

$$\frac{EI_j}{(R_j-h_j)^2}\left(w+\frac{d^2w}{d\theta^2}\right)=-M,$$

and hence from equation (29),

$$\frac{d^2w}{d\theta^2}+w=-\frac{(R_j-h_j)^2}{EI_j}(m_0+m_1\sin\theta+m_2\cos\theta+ +\sin n\theta,\cos n\theta\text{ terms}),$$

which has the general solution

$$w=-\frac{(R_j-h_j)^2}{EI_j}\left(m_0-\frac{1}{2}m_1\theta\cos\theta+ +\frac{1}{2}m_2\theta\sin\theta+\sin n\theta,\cos n\theta\text{ terms}\right)+$$

+ a complementary function of the form

$$(w_1\sin\theta+w_2\cos\theta)$$

which merely represents a "rigid-body" displacement of the frame.

Now w must be a periodic function of θ, of period 2π, which leads to the conclusion that both m_1 and m_2 must be zero. It is assumed here that the frame is initially unstrained, for otherwise a cut made at any cross section would give a discontinuity in w at that section, from the unstrained shape. Alternatively, w is defined as the radial deflection from the *unloaded* state, rather than from the *unstrained* state. The quantity M is then the bending moment due to the applied loads, and does not include any initial bending moment which may be "built in" to the frame during manufacture.

Finally, the value of m_0 may be found by consideration of the circumferential strain in the frame, along its neutral axis.

This is given by $(w+dv/d\theta)/(R_j-h_j)$, while it is also equal to T/E divided by the cross-sectional area of the frame. The cross-sectional area is most conveniently expressed here as the moment of inertia (I_j) divided by the square of the radius of gyration (ρ_j) of the section, so that

$$w+\frac{dv}{d\theta}=\frac{(R_j-h_j)\rho_j^2T}{EI_j}.$$

Now v must be a periodic function of θ, of period 2π (assuming v to be the tangential deflection from the *unloaded* state) so $dv/d\theta$ will have no constant term.

Thus, the constant term in w must be equal to the constant term in

$$\frac{(R_j-h_j)\rho_j^2T}{EI_j}.$$

whence

$$m_0=\text{constant term in}\frac{-\rho_j^2T}{(R_j-h_j)},$$

and from equation (26) this

$$=\text{constant term in}-\frac{\rho_j^2R_j\overline{N}_{Fj}}{(R_j-h_j)}.$$

Hence, from equation (20),

$$m_0=-\frac{\rho_j^2\overline{K}_{0,j}}{(R_j-h_j)} \quad . \quad . \quad . \quad (30)$$

The frame bending moment is then given by equation (29), with this value of m_0, and with $m_1=m_2=0$. In passing, it may be noted that point loads applied to a frame will give rise to a bending moment of the form $(m_1\sin\theta+m_2\cos\theta)$, and the reason why these terms are absent is that the applied loads have been assumed to be continuously distributed round the frame. Point loads may be dealt with by the present method, but only as a limiting case of a distributed loading.

A useful alternative expression for the frame bending moment, presenting a much less cumbersome appearance than equation (29), may be obtained as follows. If $P_{nj}=Q_{nj}=0$ for $n=2$, 3, etc., the bending moment in the frame will evidently be that due to the given applied loads, balanced by skin loading of the form

$$q_{Fj}R_j=(P_{1,j}\sin\theta+Q_{0,j}+Q_{1,j}\cos\theta),$$

i.e. by the loading of elementary bending and torsion theory. Denoting this bending moment by $\overline{M}$, equation (29) then takes the form

$$M=\overline{M}-\sum_{n=2}\frac{(R_j-n^2h_j)(P_{nj}\cos n\theta-Q_{nj}\sin n\theta)}{n(n^2-1)} \quad . \quad (31)$$

Equation (31) is often in practice more suitable for calculating M than equation (29), since the series for $(\overline{M}-M)$ is usually very rapidly convergent, much more so than the series of (29), while $\overline{M}$ is given by standard formulae for the usual types of loading, or may be derived by a simple strain energy calculation on a frame.

Equation (29), however, is more convenient for the calculation of the strain energy terms which are required in the present investigations.

6.4. THE FRAME END LOAD

The tension in the frame is given by equation (26). Substituting for M from (29), and for the other quantities from equations (18), (19), (20) and (21) the following equation is obtained after some simplification:

$$T=\overline{K}_{0,j}+\frac{R_j}{R_j-h_j}\{(\overline{G}_{1,j}+\overline{K}_{1,j})\cos\theta-(\overline{H}_{1,j}-\overline{J}_{1,j})\sin\theta\}+$$
$$+\sum_{n=2}\{n(P_{nj}+\overline{P}_{nj})-\overline{K}_{nj}\}\frac{\cos n\theta}{n^2-1}-$$
$$-\sum_{n=2}\{n(Q_{nj}+\overline{Q}_{nj})+\overline{J}_{nj}\}\frac{\sin n\theta}{n^2-1}. \quad (32)$$

The $n=1$ term is put in this form by using the equilibrium conditions

$$P_{1,j}+\overline{P}_{1,j}-\overline{K}_{1,j}=Q_{1,j}+\overline{Q}_{1,j}+\overline{J}_{1,j}=0$$

The alternative form for the frame end load, corresponding to equation (31), is immediately derivable from the latter with the aid of (26) and (18), and is

$$T=\overline{T}+\sum_{n=2}\frac{n}{n^2-1}(P_{nj}\cos n\theta-Q_{nj}\sin n\theta) \quad (33)$$

where $\overline{T}$ is the tension in the frame as given by the "elementary" theory.

This also follows from (32), since all the "barred" terms make up $\overline{T}$.

6.5. THE SHEAR IN THE FRAME

The shear in the frame may be similarly obtained, using either of equations (22) and (25). The $n=1$ term is again simplified by bringing in the conditions of overall equilibrium.

Leaving out intermediate steps, the required result is

$$S=\overline{H}_{0,j}+\frac{(R_j\overline{G}_{1,j}+h_j\overline{K}_{1,j})\sin\theta+(R_j\overline{H}_{1,j}-h_j\overline{J}_{1,j})\cos\theta}{R_j-h_j}+$$
$$+\sum_{n=2}(P_{nj}+\overline{P}_{nj}-n\overline{K}_{nj})\frac{\sin n\theta}{n^2-1}+\sum_{n=2}(Q_{nj}+\overline{Q}_{nj}+n\overline{J}_{nj})\frac{\cos n\theta}{n^2-1}$$
$$. \quad . \quad . \quad (34)$$

The "barred" terms make up the shear ($\overline{S}$) which would be obtained from the "elementary" theory, whence is derived the alternative form (corresponding to equations (31) and (33)).

$$S=\overline{S}+\sum_{n=2}\left(\frac{P_{nj}\sin n\theta+Q_{nj}\cos n\theta}{n^2-1}\right) \quad . \quad (35)$$

7. The Strain Energy of Bending, End Load, and Shear of Frame j

7.1. THE STRAIN ENERGY OF BENDING

Let U_{MFj} be the strain energy of bending of frame j.

Then $$6EU_{MFj}=\frac{3(R_j-h_j)}{I_j}\int_{-\pi}^{\pi}M^2\,d\theta,$$

which using (29)

$$=\frac{3(R_j-h_j)}{I_j}\Big[\frac{2\pi\rho_j{}^4\overline{K}_{0,j}{}^2}{(R_j-h_j)^2}+$$
$$+\pi\sum_{n=2}\frac{1}{n^2(n^2-1)^2}\{(R_j-n^2h_j)(P_{nj}+\overline{P}_{nj})-$$
$$-n(R_j-h_j)\overline{K}_{nj}-(n^2-1)R_j\overline{G}_{nj}\}^2+$$
$$+\pi\sum_{n=2}\frac{1}{n^2(n^2-1)^2}\{(R_j-n^2h_j)(Q_{nj}+\overline{Q}_{nj})+$$
$$+n(R_j-h_j)\overline{J}_{nj}-(n^2-1)R_j\overline{H}_{nj}\}^2\Big]$$

Therefore

$$6E\frac{\partial U_{MFj}}{\partial P_{nj}}=\frac{6\pi(R_j-h_j)(R_j-n^2h_j)}{n^2(n^2-1)^2I_j}\{(R_j-n^2h_j)(P_{nj}+\overline{P}_{nj})-$$
$$-n(R_j-h_j)\overline{K}_{nj}-(n^2-1)R_j\overline{G}_{nj}\}\quad .\quad (36)$$

and

$$6E\frac{\partial U_{MFj}}{\partial Q_{nj}}=\frac{6\pi(R_j-h_j)(R_j-n^2h_j)}{n^2(n^2-1)^2I_j}\{(R_j{}^2-n^2h_j)(Q_{nj}+\overline{Q}_{nj})+$$
$$+n(R_j-h_j)\overline{J}_{nj}-(n^2-1)R_j\overline{H}_{nj}\}\quad .\quad .\quad (37)$$

The quantities P_{nj} and Q_{nj} are defined by (18), and are linear functions of $\psi_{n,j-1}$, ψ_{nj}, $\psi_{n,j+1}$ and of $\phi_{n,j-1}$, ϕ_{nj}, $\phi_{n,j+1}$, respectively. In what follows only the ψ-terms (and their associated loading terms) will be considered in detail, since the ϕ-terms may be written down immediately from these by comparison of equations (36) and (37).

Using matrix notation

$$P_{nj}=\frac{1}{n}\Big[-\frac{R_{j-1}}{L_{j-1}},\Big(\frac{R_{j-1}}{L_{j-1}}+\frac{R_{j+1}}{L_j}\Big),-\frac{R_{j+1}}{L_j}\Big]$$
$$\{\psi_{n,j-1},\psi_{nj},\psi_{n,j+1}\}\quad .\quad (38)$$
$$(n=2,\ 3,\ \text{etc.})$$

The braces { } are used here to signify a column.

Then $$\Big\{\frac{\partial U_{MFj}}{\partial\psi_{n,j-1}},\frac{\partial U_{MFj}}{\partial\psi_{nj}},\frac{\partial U_{MFj}}{\partial\psi_{n,j+1}}\Big\}$$
$$=\frac{1}{n}\Big\{-\frac{R_{j-1}}{L_{j-1}},\Big(\frac{R_{j-1}}{L_{j-1}}+\frac{R_{j+1}}{L_j}\Big),-\frac{R_{j+1}}{L_j}\Big\}\frac{\partial U_{MFj}}{\partial P_{nj}}$$

Denoting the left-hand side of this equation by $\partial U_{MFj}/\partial\psi_n$ and using (36) and (38),

$$6E\frac{\partial U_{MFj}}{\partial\psi_n}=\frac{6\pi(R_j-h_j)(R_j-n^2h_j)^2}{n^4(n^2-1)^2I_j}$$
$$\Big\{-\frac{R_{j-1}}{L_{j-1}},\Big(\frac{R_{j-1}}{L_{j-1}}+\frac{R_{j+1}}{L_j}\Big),-\frac{R_{j+1}}{L_j}\Big\}$$
$$\Big[-\frac{R_{j-1}}{L_{j-1}},\Big(\frac{R_{j-1}}{L_{j-1}}+\frac{R_{j+1}}{L_j}\Big),-\frac{R_{j+1}}{L_j}\Big]\{\psi_{n,j-1},\psi_{nj},\psi_{n,j+1}\}$$
$$+\frac{6\pi(R_j-h_j)(R_j-n^2h_j)}{n^3(n^2-1)^2I_j}$$
$$\{(R_j-n^2h_j)\overline{P}_{nj}-n(R_j-h_j)\overline{K}_{nj}-(n^2-1)R_j\overline{G}_{nj}\}$$
$$\Big\{-\frac{R_{j-1}}{L_{j-1}},\Big(\frac{R_{j-1}}{L_{j-1}}+\frac{R_{j+1}}{L_j}\Big),-\frac{R_{j+1}}{L_j}\Big\}$$

i.e. $$6E\frac{\partial U_{MFj}}{\partial\psi_n}=\frac{6\pi(R_j-h_j)(R_j-n^2h_j)^2}{n^4(n^2-1)^2I_j}\times$$

$$\times\begin{bmatrix}\frac{R_{j-1}{}^2}{L_{j-1}{}^2}, & -\frac{R_{j-1}}{L_{j-1}}\Big(\frac{R_{j-1}}{L_{j-1}}+\frac{R_{j+1}}{L_j}\Big), & \frac{R_{j-1}R_{j+1}}{L_{j-1}L_j}\\ * , & \Big(\frac{R_{j-1}}{L_{j-1}}+\frac{R_{j+1}}{L_j}\Big)^2, & -\Big(\frac{R_{j-1}}{L_{j-1}}+\frac{R_{j+1}}{L_j}\Big)\frac{R_{j+1}}{L_j}\\ * , & * , & \frac{R_{j+1}{}^2}{L_j{}^2}\end{bmatrix}\begin{bmatrix}\psi_{n,j-1}\\ \psi_{nj}\\ \psi_{n,j+1}\end{bmatrix}$$
$$(\textit{Structural Terms})$$
$$+\frac{6\pi(R_j-h_j)(R_j-n^2h_j)}{n^3(n^2-1)^2I_j}\{(R_j-n^2h_j)\overline{P}_{nj}$$
$$-n(R_j-h_j)\overline{K}_{nj}-(n^2-1)R_j\overline{G}_{nj}\}\begin{bmatrix}-\frac{R_{j-1}}{L_{j-1}}\\ \Big(\frac{R_{j-1}}{L_{j-1}}+\frac{R_{j+1}}{L_j}\Big)\\ -\frac{R_{j+1}}{L_j}\end{bmatrix}$$
$$(\textit{Loading Terms})$$
$$.\quad .\quad .\quad .\quad (39)$$

The asterisks in the square matrix indicate that the matrix is symmetrical about its leading diagonal.

The complete set of partial differential coefficients for frame j is given by equation (39), taking n successively equal to 2, 3, ... etc.

7.2. THE STRAIN ENERGY OF END LOAD

Let U_{TFj} denote the strain energy of end load in frame j.

Then $$6EU_{TFj}=\frac{3(R_j-h_j)\rho_j{}^2}{I_j}\int_{-\pi}^{\pi}T^2d\theta,$$

which, using (32)

$$=\frac{3\pi(R_j-h_j)\rho_j{}^2}{I_j}\sum_{n=2}\frac{(nP_{nj}+n\overline{P}_{nj}-\overline{K}_{nj})^2+(nQ_{nj}+n\overline{Q}_{nj}+\overline{J}_{nj})^2}{(n^2-1)^2}$$
$$+n=0,\ 1\ \text{terms.}$$

Therefore

$$6E\frac{\partial U_{TFj}}{\partial P_{nj}}=\frac{6\pi(R_j-h_j)\rho_j{}^2(n^2P_{nj}+n^2\overline{P}_{nj}-n\overline{K}_{nj})}{(n^2-1)^2I_j}$$
$$.\quad .\quad .\quad .\quad (40)$$

and

$$6E\frac{\partial U_{TFj}}{\partial Q_{nj}}=\frac{6\pi(R_j-h_j)\rho_j{}^2(n^2Q_{nj}+n^2\overline{Q}_{nj}+n\overline{J}_{nj})}{(n^2-1)^2I_j}$$
$$.\quad .\quad .\quad (41)$$
$$(\text{for } n>1)$$

As before, attention is confined to the ψ-terms, *i.e.* to equation (40), whence with the aid of (38) this result is derived:

$$6E\frac{\partial U_{TFj}}{\partial\psi_n}=\frac{6\pi(R_j-h_j)\rho_j{}^2}{(n^2-1)^2I_j}\Big\{-\frac{R_{j-1}}{L_{j-1}},\Big(\frac{R_{j-1}}{L_{j-1}}+\frac{R_{j+1}}{L_j}\Big),$$
$$-\frac{R_{j+1}}{L_j}\Big\}\Big[-\frac{R_{j-1}}{L_{j-1}},\ \text{etc.}\ldots.\Big]\{\psi_{n,j-1},\psi_{nj},\psi_{n,j+1}\}$$
$$(\textit{Structural Terms})$$
$$+\frac{6\pi(R_j-h_j)\rho_j{}^2(n\overline{P}_{nj}-\overline{K}_{nj})}{(n^2-1)^2I_j}\Big\{-\frac{R_{j-1}}{L_{j-1}},\Big(\frac{R_{j-1}}{L_{j-1}}+\frac{R_{j+1}}{L_j}\Big),$$
$$-\frac{R_{j-1}}{L_j}\Big\}.\quad .\quad (42)$$
$$(\textit{Loading Terms})$$

This equation may be written in the same way as equation (39). Clearly the only difference is in the values of the coefficients in front of the matrices, and to obtain the strain energy of bending and end load the corresponding coefficients of (39) and (42) are simply added together.

In passing it should be noted that the "simple beam theory" expressions were used for the values of U_{Fj} and U_{TFj}. These are good enough if the radius (R_j) is large compared with the depth of the cross section, and are probably acceptable for most of the frames likely to be encountered in practice.

7.3. THE SHEAR STRAIN ENERGY

Let U_{SFj} denote the shear strain energy in frame j. This is the strain energy in the web of the frame. It is assumed here that the cross section of the frame consists essentially of two flanges which carry the end load and bending moment, connected by a web for taking the shear. The mean shear loading in the web will be equal to S, divided by the depth between the centroids of the two flanges.

Then the element of strain energy in the web is approximately

$$\frac{1}{2C}\left(\frac{S}{d_j t_{wj}}\right)^2 t_{wj} d_{wj} (R_j - h_j)\, d\theta,$$

taking $(R_j - h_j)$ to be approximately the mean radius,

i.e. $$dU_{SFj} = \frac{S^2 d_{wj} (R_j - h_j)\, d\theta}{2C t_{wj} d_j^2}$$

The quantity $d_j^2 t_{wj}/d_{wj}$ has the dimensions of an area, and may be regarded as the effective web cross-sectional area for the purpose of strain energy calculation. For the idealised cross section consisting of two point flanges joined by a web, it is equal to the actual web cross section, since then $d_{wj} = d_j$.

It is convenient to denote the ratio of the *total* flange area ($= I_j/\rho_j^2$) to the effective web area by the symbol η_j.

i.e. $$\frac{t_{wj} d_j^2}{d_{wj}} = \frac{I_j}{\rho_j^2 \eta_j}.$$

Then $$dU_{SFj} = \frac{S^2 \rho_j^2 \eta_j (R_j - h_j)\, d\theta}{2CI_j}$$

Therefore

$$6EU_{SFj} = \frac{3E\rho_j^2 \eta_j (R_j - h_j)}{CI_j} \int_{-\pi}^{\pi} S^2 d\theta,$$

which from (34)

$$= \frac{3\pi E \rho_j^2 \eta_j (R_j - h_j)}{CI_j} \sum_{n=2} \frac{(P_{nj} + \bar{P}_{nj} - n\bar{K}_{nj})^2 + (Q_{nj} + \bar{Q}_{nj} + n\bar{J}_{nj})^2}{(n^2-1)^2}$$

$$+ n = 0,\ 1 \text{ terms.}$$

Therefore

$$6E \frac{\partial U_{SFj}}{\partial P_{nj}} = \frac{6\pi E \rho_j^2 \eta_j (R_j - h_j)(P_{nj} + \bar{P}_{nj} - n\bar{K}_{nj})}{CI_j (n^2-1)^2} \quad (43)$$

and

$$6E \frac{\partial U_{SFj}}{\partial Q_{nj}} = \frac{6\pi E \rho_j^2 \eta_j (R_j - h_j)(Q_{nj} + \bar{Q}_{nj} + n\bar{J}_{nj})}{CI_j (n^2-1)^2} \quad (44)$$

Again confining attention to the ψ-terms, the following equation is obtained from equations (43) and (38)

$$6E \frac{\partial U_{SFj}}{\partial \psi_n} = \frac{6\pi E \rho_j^2 \eta_j (R_j - h_j)}{CI_j n^2 (n^2-1)^2} \left\{ -\frac{R_{j-1}}{L_{j-1}},\ \text{etc.} \right\} \left[-\frac{R_{j-1}}{L_{j-1}},\ \text{etc.} \right] \{\psi_{n,j-1}, \psi_{nj}, \psi_{n,j+1}\}$$

(*Structural Terms*)

$$+ \frac{6\pi E \rho_j^2 \eta_j (R_j - h_j)(\bar{P}_{nj} - n\bar{K}_{nj})}{CI_j n (n^2-1)^2} \left\{ -\frac{R_{j-1}}{L_{j-1}},\ \text{etc.} \right\}. \quad (45)$$

(*Loading Terms*)

A similar expression, for $6E\ \partial U_{SFj}/\partial \phi_n$, derives from (44). As with equation (42), equation (45), when written out more extensively, will take the same form as (39) but with different coefficients in front of the matrices.

7.4. THE TOTAL STRAIN ENERGY OF THE FRAME

Let U_{Fj} denote the total strain energy of frame j, so that

$$U_{Fj} = U_{MFj} + U_{TFj} + U_{SFj}.$$

Then the expression for $6E\ \partial U_{Fj}/\partial \psi_n$ is obtained by the addition of equations (39), (42) and (45). writing the last two of these in the same form as (39). It is convenient for purposes of numerical computation to put the coefficients in front of the matrices in the form $6\pi (R_j - h_j)/[(n^2-1)^2 I_j]$ multiplied by a polynomial in $1/n$, so that finally

$$6E \frac{\partial U_{Fj}}{\partial \psi_n} =$$

$$\frac{6\pi (R_j - h_j)}{(n^2-1)^2 I_j} \left\{ \frac{R_j^2}{n^4} - \frac{1}{n^2}\left(2R_j h_j - \frac{E\rho_j^2 \eta_j}{C}\right) + h_j^2 + \rho_j^2 \right\}$$

$$\begin{bmatrix} \frac{R_{j-1}^2}{L_{j-1}^2}, & -\frac{R_{j-1}}{L_{j-1}}\left(\frac{R_{j-1}}{L_{j-1}} + \frac{R_{j+1}}{L_j}\right), & \frac{R_{j-1} R_{j+1}}{L_{j-1} L_j} \\ * , & \left(\frac{R_{j-1}}{L_{j-1}} + \frac{R_{j+1}}{L_j}\right)^2, & -\left(\frac{R_{j-1}}{L_{j-1}} + \frac{R_{j+1}}{L_j}\right)\frac{R_{j+1}}{L_j} \\ * , & * , & \frac{R_{j+1}^2}{L_j^2} \end{bmatrix} \begin{bmatrix} \psi_{n,j-1} \\ \psi_{nj} \\ \psi_{n,j+1} \end{bmatrix}$$

(*Structural Terms*)

+

$$\frac{6\pi (R_j - h_j)}{(n^2-1)^2 I_j} \Big[n\bar{P}_{nj} \left\{ \frac{R_j^2}{n^4} - \frac{1}{n^2}\left(2R_j h_j - \frac{E\rho_j^2 \eta_j}{C}\right) + h_j^2 + \rho_j^2 \right\} -$$

$$- \bar{K}_{nj} \left\{ \frac{R_j (R_j - h_j)}{n^2} - h_j (R_j - h_j) + \rho_j^2 + \frac{E\rho_j^2 \eta_j}{C} \right\}$$

$$- nR_j \bar{G}_{nj} \left(\frac{R_j}{n^4} - \frac{h_j}{n^2}\right)(n^2-1) \Big] \begin{bmatrix} -\frac{R_{j-1}}{L_{j-1}} \\ \left(\frac{R_{j-1}}{L_{j-1}} + \frac{R_{j+1}}{L_j}\right) \\ -\frac{R_{j+1}}{L_j} \end{bmatrix}$$

(*Loading Terms*)

. . . (46)

($n = 2,\ 3,$ etc.)

Equation (46) is necessarily somewhat cumbersome in appearance, but it will be seen later that the computation of the various coefficients can be largely reduced to a matter of routine by suitable tabulation.

It will often happen in practice that the externally applied tangential loads are put into the frame along its neutral axis. When this is the case the coefficient for the loading terms simplifies to some extent, as follows.

Referring to Fig. 3 and considering the piece of frame between the radii θ and $\theta+\delta\theta$,

$$(\bar{q}_{Fj}R_j\delta\theta)\,h_j+\bar{M}_{Fj}R_j\delta\theta=0,$$

the moment of the applied loads about the neutral axis being zero.

Thus, $$\bar{q}_{Fj}h_j+\bar{M}_{Fj}=0,$$

or in the present case

$$\frac{\bar{P}_{nj}h_j}{R_j}+\bar{G}_{nj}=0.$$

Eliminating $\bar{G}_{nj}$, the loading coefficient then reduces to

$$\frac{6\pi(R_i-h_i)}{(n^2-1)^2\,I_j}\Big[n\bar{P}_{nj}\left\{\frac{R_j(R_j-h_j)}{n^4}-\right.$$

$$\left.-\frac{1}{n^2}\left(R_jh_j-h_j^2-\frac{E\rho_j^2\eta_j}{C}\right)+\rho_j^2\right\}$$

$$-\bar{K}_{nj}\left\{\frac{R_j(R_j-h_j)}{n^2}-h_j(R_j-h_j)+\rho_j^2+\frac{E\rho_j^2\eta_j}{C}\right\}\Big] \qquad (47)$$

To conclude this section, the expression for $6E\partial U_{Fj}/\partial\phi_n$ may be immediately deduced from equation (46) by replacing the ψ's in the latter by the corresponding ϕ's, and by replacing $\bar{P}_{nj}$ by $\bar{Q}_{nj}$, $\bar{K}_{nj}$ by $-\bar{J}_{nj}$, and $\bar{G}_{nj}$ by $\bar{H}_{nj}$. The alterations in the loading coefficient are obvious by inspection of equations (36), (37), (40), (41), (43) and (44).

8. Loading Terms for Various Types of Applied Loading

8.1. RADIAL LOAD $+W$ AT $\theta=0$ (Case A)

A radial load at $\theta=0$ may be regarded as the limiting case of a radial loading distributed over an arc, centred on $\theta=0$, when the length of the arc tends to zero, keeping the total load finite. For this purpose, consider the "triangular" loading defined by the equation

$$\bar{N}_{Fj}R_j=\frac{W}{\beta}\left(1-\frac{\theta}{\beta}\right)$$

between $|\theta|=0$ and β (radians), being zero elsewhere.

The total radial load is equal to W, while the requirement that $d\bar{N}_{Fj}/d\theta$ must be everywhere finite (since it enters into equation (27)) is satisfied except in the limit $\beta\longrightarrow 0$. As a matter of interest this requirement appears to be an unnecessary restriction, since as far as $\bar{N}_{Fj}$ is concerned equation (27) may be immediately integrated with respect to θ. In fact, a concentrated radial load could be treated as the limit of a load *uniformly* distributed over the arc $-\beta\leqslant\theta\leqslant\beta$, as was done in Ref. 1. The final result must, of course, be the same.

Returning to the present proposal, $\bar{N}_{Fj}R_j$ now has to be expressed as a series of the form $\sum_{n=0}\bar{K}_{nj}\cos n\theta$ (*see* equation (20)).

Thus

$$\bar{K}_{0,j}=\frac{1}{\pi}\int_0^{\pi}\bar{N}_{Fj}R_j\,d\theta=\frac{W}{\pi\beta}\int_0^{\beta}\left(1-\frac{\theta}{\beta}\right)d\theta=\frac{W}{2\pi},$$

and $$\bar{K}_{nj}=\frac{2}{\pi}\int_0^{\pi}\bar{N}_{Fj}R_j\cos n\theta\,d\theta,\ (n>0)$$

$$=\frac{2W}{\pi\beta}\int_0^{\beta}\left(1-\frac{\theta}{\beta}\right)\cos n\theta\,d\theta$$

$$=\frac{2W(1-\cos n\beta)}{\pi n^2\beta^2}\longrightarrow\frac{W}{\pi}\text{ as }n\beta\longrightarrow 0.$$

With a *fixed* value of β, however small, $n\beta\longrightarrow\infty$ as $n\longrightarrow\infty$, and then $\bar{K}_{nj}\longrightarrow 0$. In any case, however, the loading terms in equation (39) due to $\bar{K}_{nj}$ tend to zero as $n\longrightarrow\infty$, due to the factor $(n^2-1)^2$ in the denominator. Moreover, the structural terms for the frames (equation (39)) and the skin (equation (9)) tend to zero as $n\longrightarrow\infty$, but, since the structural terms for the stringers (equation (16)) are independent of n, the coefficients of the ψ_{nj}'s remain finite. Hence $\psi_{nj}\longrightarrow 0$ as $n\longrightarrow\infty$, and so, for a radial load W at $\theta=0$,

$$\bar{K}_{0,j}=\frac{W}{2\pi},\ K_{nj}(n>0)=\frac{W}{\pi}. \qquad (48)$$

8.2. RADIAL LOADS $+\frac{1}{2}W$ AT $\theta=\pm\gamma$

This may be deduced from case A, since

$$\bar{N}_{Fj}R_j=\left\{\frac{W}{4\pi}+\frac{W}{2\pi}\sum_{n=1}\cos n(\theta-\gamma)\right\},\text{ due to }\frac{W}{2}\text{ at }\theta=\gamma,\ +$$

$$+\left\{\frac{W}{4\pi}+\frac{W}{2\pi}\sum_{n=1}\cos n(\theta+\gamma)\right\},\text{ due to }\frac{W}{2}\text{ at }\theta=-\gamma,$$

$$=\frac{W}{2\pi}+\frac{W}{\pi}\sum_{n=1}\cos n\gamma\cos n\theta.$$

Therefore

$$\bar{K}_{0,j}=\frac{W}{2\pi},\ \bar{K}_{nj}(n>0)=\frac{W\cos n\gamma}{\pi} \qquad (49)$$

From (48) and (49) we immediately deduce case C.

8.3. RADIAL LOADS $+W$ AT $\theta=0$ AND $-W$ AT $\theta=\pi$ (Case C)

This is case A − case B with $\gamma=\pi$.

Hence

$$\bar{K}_{0,j}=0,\ \bar{K}_{nj}(n>0)=\frac{W}{\pi}(1-\cos n\pi)=\frac{2W\sin^2\frac{n\pi}{2}}{\pi} \qquad (50)$$

Thus $\bar{K}_{n,j}=0$ if n is even, and $=2W/\pi$ if n is odd. Note that for this case both loads would be "downwards" in Fig. 2.

8.4. RADIAL LOADS $+W$ AT $\theta=\gamma$ AND $-W$ AT $\theta=-\gamma$ (Case D)

This loading may be expressed by a series of the type

$\overline{N}_{Fj}R_j=\sum_{n=1}\overline{J}_{nj}\sin n\theta$ (equation (20)), from which

$$\overline{J}_{nj}=\frac{2}{\pi}\int_0^{\pi}\overline{N}_{Fj}R_j\sin n\theta d\theta=\frac{2W\sin n\gamma}{\pi}. \qquad (51)$$

This result may be obtained by a limiting process similar to that employed for case A, but is very simply, if somewhat uncritically, deduced from the consideration that, since all the load is concentrated at $\theta=\gamma$, the value of the integral is

$$\sin n\gamma\int_0^{\pi}\overline{N}_{Fj}R_j d\theta=W\sin n\gamma.$$

8.5. TANGENTIAL LOAD $+W$ AT $\theta=0$ (Case E)

For this case take

$$\overline{q}_{Fj}R_j=\sum_{n=0}\overline{Q}_{nj}\cos n\theta,$$

and by comparison with case A it is seen that

$$\overline{Q}_{0,j}=\frac{W}{2\pi},\ \overline{Q}_{nj}\,(n>0)=\frac{W}{\pi} \quad . \quad . \qquad (52)$$

8.6. TANGENTIAL LOADS $+\frac{1}{2}W$ AT $\theta=\pm\gamma$ (Case F)

By comparison with case B

$$\overline{Q}_{0,j}=\frac{W}{2\pi},\ \overline{Q}_{nj}\,(n>0=\frac{W\cos n\gamma}{\pi} \quad . \qquad (53)$$

With $\gamma=\pi/2$ the loads are equivalent to a torque of amount WR, and $\overline{Q}_{nj}\ (n>0)=W\cos(n\pi/2)/\pi$, which is zero when n is odd and equal to $(-1)^{n/2}\ W/\pi$ when n is even.

8.7. TANGENTIAL LOADS $+W$ AT $\theta=\gamma$ and $-W$ AT $\theta=-\gamma$ (Case G)

For this case take

$$\overline{q}_{Fj}R_j=\sum_{n=1}\overline{P}_{nj}\sin n\theta,$$

then from case D,

$$\overline{P}_{nj}=\frac{2W\sin n\gamma}{\pi} \quad . \quad . \quad . \qquad (54)$$

For $\gamma=\pi/2$ this becomes $\overline{P}_{nj}=2W\sin(n\pi/2)/\pi$, which is zero when n is even and equal to $(-1)^{\frac{1}{2}(n-1)}\ 2W/\pi$ when n is odd.

Cases A to G cover the principal types of applied loading that are likely to be met with in practice. Cases of concentrated applied moments may be treated similarly, and it is not necessary to go into detail here.

9. The Strain Energy Equations

Expressions have now been obtained for the derivatives of the strain energy of each part of the structure in terms of the various unknowns, so that the formation of the strain energy equations can now be discussed. Let U_j denote the strain energy of bay j (including frame j), and U the strain energy of the complete structure.

Thus,

$$U_j=U_{Sj}+U_{Nj}+U_{Fj}, \text{ and } U=\sum U_j.$$

Assume that j takes the values 0, 1, 2, ... J, $J+1$, the end frames being numbered 0 and $J+1$. The unknowns are then the ψ_{nj}'s for $j=1, 2, \ldots . J$; $n=2, 3$, etc.

The remaining ψ's ($\psi_{0,j}$, $\psi_{1,j}$ for $j=1$ to J and $\psi_{n,0}$, $\psi_{n,J+1}$ for $n=0, 1, 2, 3$, etc.) are known quantities, since the total end load and bending moment at every cross section, and the distribution of load on the end cross sections, are known. The case of a "rigidly built-in" end cross section will not be considered here since it is hardly likely to arise in practice.

Similar remarks apply to the ϕ-system of variables, and since in fact the whole of the following discussion will apply equally to the ϕ- or ψ-systems only the latter will be discussed as in the previous sections. The modifications to the loading terms in passing from one system of variables to the other are trivial, and are referred to at the end of Section 7.

To continue

$$6E\frac{\partial U_j}{\partial\psi_n}=6E\frac{\partial U_{Sj}}{\partial\psi_n}(\text{eqn. } 9)+$$
$$+6E\frac{\partial U_{Nj}}{\partial\psi_n}(\text{eqn. } 16)+6E\frac{\partial U_{Fj}}{\partial\psi_n}(\text{eqn. } 46) \quad . \qquad (55)$$

Referring to the equations mentioned here, the most important point to note is that the strain energy derivative with respect to any particular ψ contains only ψ's having the same value of n. This is a consequence of the representation of the stress distributions in the form of trigonometrical series, and of the uniformity of the structure in the circumferential direction. For example, the derivative $6E\ \partial U_j/\partial\psi_{nj}$, which is obtained by taking the top row of (9), the top row of (16) and the middle row of (46), contains $\psi_{n,j-1}$, ψ_{nj}, $\psi_{n,j+1}$ and terms depending on the applied loads.

The typical strain energy equation is

$$6E\frac{\partial U}{\partial\psi_{nj}}=0, \quad . \quad . \quad . \qquad (56)$$

and, in view of what has just been said, it is seen that

$$6E\frac{\partial U}{\partial\psi_{nj}}=\frac{6E\partial U_{j-1}}{\partial\psi_{nj}}+\frac{6E\partial U_j}{\partial\psi_{nj}}+\frac{6E\partial U_{j+1}}{\partial\psi_{nj}}$$

(in general, but some of these items will be missing at each end of the structure.) The expression $6E\,\partial U_{j-1}/\partial\psi_{nj}$ is derived from equations (9), (16) and (46) by replacing j by $j-1$, and taking the sum (bottom row of (9) + bottom row of (16) + bottom row of (46)), while the expression $6E\partial U_{j-1}/\partial\xi_{nj}$ derives from equation (46) only, being given by the top row of the latter after replacing j by $j+1$.

When all the various items are added together a relation is obtained between five consecutive ψ's (*i.e.* five consecutive values of j) which may be conveniently represented in the form

$$6E\frac{\partial U}{\partial\psi_{nj}}=c_{n,j-2}\psi_{n,j-2}+b_{n,j-1}\psi_{n,j-1}+$$
$$+a_{nj}\psi_{nj}+b_{nj}\psi_{n,j+1}+c_{nj}\psi_{n,j+2}+l_{nj}=0 \quad . \qquad (57)$$

The first five terms in this equation are the structural terms, and the relationships between the coefficients of

$\psi_{n,j-1}$ and $\psi_{n,j+1}$, and of $\psi_{n,j-2}$ and $\psi_{n,j+2}$, follow from the symmetry of the matrix of the structural coefficients of the complete set of equations about its leading diagonal. This, in turn, is due to the circumstance that the strain energy (U) is a quadratic function of the variables.

NOTE ON THE END CONDITIONS

Equation (57) requires further consideration when $j=1$, 2, $N-1$ or N, since in these cases the loading applied to the end cross sections ($j=0$ and $N+1$) must be taken into account. The loading coefficient l_{nj} contains only rib loading terms, being derived entirely from equation (46), applied to the three consecutive ribs $j-1$, j and $j+1$.

It is clearly sufficient to consider only the end $j=0$, and putting $j=1$ in equation (57) the two left-hand terms become $c_{n,-1}\,\psi_{n,-1}+b_{n,0}\,\psi_{n,0}$, while for $j=2$ the left-hand term of (57) is $c_{n,0}\,\psi_{n,0}$.

The value of $\psi_{n,0}$ is known, but it is not clear what significance should be attached to $\psi_{n,-1}$, since there is no "bay -1." $\psi_{n,-1}$ may be regarded as a coefficient of the direct stress loading ($N_{j,\,j=-1}$) at the remote end of a fictitious "bay -1" which is so long that it can have no constraining effect on the cross section at frame 0, *i.e.* its length $L_{-1}=\infty$. Reference to equation (46) applied to rib 0, then shows that $c_{n,-1}=0$, c_{nj} being the coefficient of $\psi_{n,j+2}$ in the top row of (46) (applied to rib $j+1$). Thus, for $j=1$ the two left-hand terms of equation (57) reduce to $b_{n,0}\,\psi_{n,0}$ (note that $b_{n,0}$ will be affected by the assumption that $L_{-1}=\infty$), and the equation then becomes

$$a_{n,1}\,\psi_{n,1}+b_{n,1}\,\psi_{n,2}+c_{n,1}\,\psi_{n,3}+(l_{n,1}+b_{n,0}\,\psi_{n,0})=0,$$

the item $b_{n,0}\,\psi_{n,0}$ in effect constituting an addition to the loading term.

Similarly, taking $j=2$, equation (57) may be put in the form

$$b_{n,1}\,\psi_{n,1}+a_{n,2}\,\psi_{n,2}+b_{n,2}\,\psi_{n,3}+c_{n,2}\psi_{n,4}+(l_{n,2}+c_{n,0}\,\psi_{n,0})=0.$$

A particularly simple case arises when $\psi_{n,0}=0$ ($n>1$), that is, when there is no direct stress loading on the end cross section, or when such loading is of the "simple beam-theory" type.

10. The Computation of the Strain Energy Coefficients a_{nj}, b_{nj}, c_{nj} and l_{nj}

10.1. THE STRUCTURAL COEFFICIENTS a_{nj}, b_{nj} AND c_{nj}

The quantity a_{nj} is the coefficient of ψ_{nj} in the equation $6E\,\partial U/\partial\psi_{nj}=0$, which will be called "strain energy equation nj."

The left-hand side of this equation is made up of items from bays $j-1$, j and $j+1$, each bay contributing one or more terms which may be obtained from equations (9), (16) and (46). Thus, it is found that a_{nj} contains skin and stringer terms from bays $j-1$ and j, and terms from frames $j-1$, j and $j+1$.

Similarly, b_{nj} contains skin and stringer terms from bay j, and terms from frames j and $j+1$, while c_{nj} is a single term from frame $j+1$.

The algebraic expressions for a_{nj} and b_{nj} are evidently very complicated, particularly when the strain energy of end load and shear in the frames is taken into account. It can be imagined that the numerical computation of these coefficients directly from such formulae could be a very troublesome business unless tackled systematically, bearing in mind that values are required over the range $j=1$ to J, and for $n=2$, 3, etc. (the highest value of n will depend on the type of loading and the degree of accuracy required). To this end, a method of tabulation has been devised which reduces the work largely to a

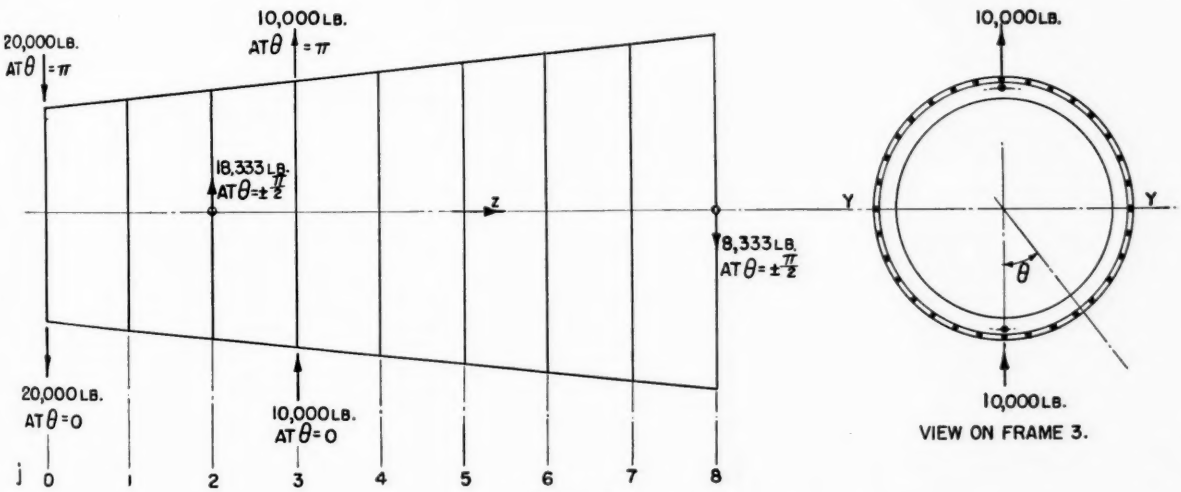


FIGURE 4.

BASIC DATA: Skin thickness = 0·048 in. throughout, E/C taken = 2·5. No. of stringers = 32; effective stringer cross section (including 1·5 in. of skin) = 0·144 sq. in., giving $A_j = 4{\cdot}608$ in.^{2}. Tangential loads (at frames 2 and 8) applied on neutral axes of frames (see expression 47) $\psi_{1,j}$ = B.M. about axis YY $\div\,\pi R_j$ = (0, −9,402, −17,462, −16,977, −12,732, −8,988, −5,659, −2,681, 0) for $j=0$ to 8. ($\psi_{0,j}=\phi_{nj}=0$). For other basic data, see Table I.

TABLE I.

THE STRUCTURAL COEFFICIENTS a_{nj}, b_{nj} and c_{nj}.

	Row No.	Item or operation	$j=0$	1	2	3	4	5	6	7	8	Row No.
BASIC DATA	(1)	R_j (inches)	30·0	32·5	35·0	37·5	40·0	42·5	45·0	47·5	50·0	(1)
	(2)	L_j (inches)	24·0	24·0	24·0	24·0	24·0	24·0	24·0	24·0	($L_{-1}=L_8=\infty$)	(2)
	(3)	$\frac{R_{j-1}}{L_{j-1}}$	0	1·25	1·3542	1·4583	1·5625	1·6667	1·7708	1·8750	1·9792	(3)
	(4)	$\frac{R_{j+1}}{L_j}$	1·3542	1·4583	1·5625	1·6667	1·7708	1·8750	1·9792	2·0833	0	(4)
	(5)	h_j (inches)	3·5	2·7	2·9	2·9	2·7	2·7	2·7	2·7	2·8	(5)
	(6)	R_j-h_j (inches)	26·5	29·8	32·1	34·6	37·3	39·8	42·3	44·8	47·2	(6)
	(7)	I_j (in.^{4})	9·0	2·5	5·0	5·0	2·5	2·5	2·5	2·5	3·75	(7)
	(8)	$\frac{6\pi(R_j-h_j)}{I_j}$ (in.$^{-3}$)	55·5	224·7	121·0	130·4	281·2	300·1	318·9	337·8	237·3	(8)
	(9)	ρ_j (inches)	3·0	2·5	2·5	2·5	2·5	2·5	2·5	2·5	2·5	(9)
	(10)	η_j (SECTION 7.3)	2·841	1·852	2·778	2·778	1·852	1·852	1·852	1·852	2·778	(10)
	(11)	$\frac{E}{C}\rho_j^2\eta_j$ (in.^{2})	63·9	28·9	43·4	43·4	28·9	28·9	28·9	28·9	43·4	(11)
RIBS	(12)	$(8)\times R_j^2$	49,950	237,340	148,225	183,375	449,920	542,060	645,770	762,160	593,250	(12)
	(13)	$(8)\times\left(2R_jh_j-\frac{E}{C}\rho_j^2\eta_j\right)$	8,109	32,941	19,312	22,703	52,613	60,200	68,276	76,883	56,145	(13)
	(14)	$(8)\times(h_j^2+\rho_j^2)$	1,179	3,042	1,774	1,912	3,807	4,063	4,318	4,574	3,344	(14)
	(15)	$(12)\times\frac{R_{j-1}^2}{L_{j-1}^2}\times10^{-6}$	0	0·3708	0·2718	0·3900	1·0984	1·5058	2·0250	2·6795	2·3239	(15)
	(16)	$(12)\times\left(\frac{R_{j-1}}{L_{j-1}}+\frac{R_{j+1}}{L_j}\right)^2\times10^{-6}$	0·0916	1·7409	1·2610	1·7908	4·9990	6·7994	9·0812	11·9416	2·3239	(16)
	(17)	$(12)\times\frac{R_{j+1}^2}{L_j^2}\times10^{-6}$	0·0916	0·5047	0·3619	0·5094	1·4108	1·9057	2·5296	3·3079	0	(17)
	(18)	"Diagonal sum" of (15), (16) & (17)	0·4624	2·1043 ←	2·1557	3·2511	7·0142	10·2352	13·6664	16·7951	5·6318	(18)
	(19)	$(13)\times\frac{R_{j-1}^2}{L_{j-1}^2}\times10^{-6}$	0	0·0515	0·0354	0·0483	0·1285	0·1672	0·2141	0·2703	0·2199	(19)
	(20)	$(13)\times\left(\frac{R_{j-1}}{L_{j-1}}+\frac{R_{j+1}}{L_j}\right)^2\times10^{-6}$	0·0149	0·2416	0·1643	0·2217	0·5846	0·7551	0·9601	1·2046	0·2199	(20)
	(21)	$(13)\times\frac{R_{j+1}^2}{L_j^2}\times10^{-6}$	0·0149	0·0701	0·0471	0·0631	0·1650	0·2116	0·2675	0·3337	0	(21)
	(22)	"Diagonal sum" of (19), (20) & (21)	0·0664	0·2919	0·2827	0·3973	0·8149	1·1342	1·4420	1·6920	0·5536	(22)
	(23)	$(14)\times\frac{R_{j-1}^2}{L_{j-1}^2}\times10^{-6}$	0	0·0048	0·0033	0·0041	0·0093	0·0113	0·0135	0·0161	0·0131	(23)
	(24)	$(14)\times\left(\frac{R_{j-1}}{L_{j-1}}+\frac{R_{j+1}}{L_j}\right)^2\times10^{-6}$	0·0022	0·0223	0·0151	0·0187	0·0423	0·0510	0·0607	0·0717	0·0131	(24)
	(25)	$(14)\times\frac{R_{j+1}^2}{L_j^2}\times10^{-6}$	0·0022	0·0065	0·0043	0·0053	0·0119	0·0143	0·0169	0·0199	0	(25)
	(26)	"Diagonal sum" of (23), (24) & (25)	0·0070	0·0278	0·0257	0·0323	0·0589	0·0764	0·0911	0·1017	0·0330	(26)

TABLE I (*Continued*)

	(26)	"Diagonal sum" of (23), (24) & (25)	0·0070	0·0278	0·0257	0·0323	0·0589	0·0764	0·0911	0·1017	0·0330	(26)

TABLE 1 (*Continued*)

RIBS (*Contd.*)	(27)	$(12)\times\left(\frac{R_{j-1}}{L_{j-1}}+\frac{R_{j+1}}{L_j}\right)\frac{R_{j-1}}{L_{j-1}}\times 10^{-6}$	0	0·8035	0·5855	0·8357	2·3433	3·1997	4·2883	5·6566	2·3239	(27)
	(28)	$(12)\times\left(\frac{R_{j-1}}{L_{j-1}}+\frac{R_{j+1}}{L_j}\right)\frac{R_{j+1}}{L_j}\times 10^{-6}$	0·0916	0·9374	0·6755	0·9551	2·6557	3·5996	4·7929	6·2850	0	(28)
	(29)	"Diagonal sum" of (27) & (28)	0·8951	1·5229	1·5112	3·2984	5·8554	7·8879	10·4495	8·6089		(29)
	(30)	$(13)\times\left(\frac{R_{j-1}}{L_{j-1}}+\frac{R_{j+1}}{L_j}\right)\frac{R_{j-1}}{L_{j-1}}\times 10^{-6}$	0	0·1115	0·0763	0·1035	0·2740	0·3553	0·4534	0·5706	0·2199	(30)
	(31)	$(13)\times\left(\frac{R_{j-1}}{L_{j-1}}+\frac{R_{j+1}}{L_j}\right)\frac{R_{j+1}}{L_j}\times 10^{-6}$	0·0149	0·1301	0·0880	0·1182	0·3106	0·3998	0·5067	0·6340	0	(31)
	(32)	"Diagonal sum" of (30) & (31)	0·1264	0·2064	0·1915	0·3922	0·6659	0·8532	1·0773	0·8539		(32)
	(33)	$(14)\times\left(\frac{R_{j-1}}{L_{j-1}}+\frac{R_{j+1}}{L_j}\right)\frac{R_{j-1}}{L_{j-1}}\times 10^{-6}$	0	0·0103	0·0070	0·0087	0·0198	0·0240	0·0287	0·0339	0·0131	(33)
	(34)	$(14)\times\left(\frac{R_{j-1}}{L_{j-1}}+\frac{R_{j+1}}{L_j}\right)\frac{R_{j+1}}{L_j}\times 10^{-6}$	0·0022	0·0120	0·0081	0·0100	0·0225	0·0270	0·0320	0·0377	0	(34)
	(35)	"Diagonal sum" of (33) & (34)	0·0125	0·0190	0·0168	0·0298	0·0465	0·0557	0·0659	0·0508		(35)
	(36)	$(12)\times\frac{R_{j-1}R_{j+1}}{L_{j-1}\ L_j}\times 10^{-6}$	0	0·4326	0·3136	0·4457	1·2449	1·6940	2·2633	2·9771		(36)
	(37)	$(13)\times\frac{R_{j-1}R_{j+1}}{L_{j-1}\ L_j}\times 10^{-6}$	0	0·0600	0·0409	0·0552	0·1456	0·1881	0·2393	0·3003		(37)
	(38)	$(14)\times\frac{R_{j-1}R_{j+1}}{L_{j-1}\ L_j}\times 10^{-6}$	0	0·0055	0·0038	0·0046	0·0105	0·0127	0·0151	0·0179		(38)
SKIN	(39)	$\frac{3\,E\,\pi\sec\alpha\,(R_j+R_{j+1})}{Ct_jL_j}$	1,285	1,388	1,491	1,594	1,697	1,799	1,902	2,005	$t_j=0{\cdot}048$ in. $\sec\alpha=1{\cdot}0054$	(39)
	(40)	"Sums in pairs" of (39)	1,285	2,673	2,879	3,085	3,291	3,496	3,701	3,907	2,005	(40)
STRINGERS	(41)	$\frac{2\pi^2 L_j\sec^3\alpha}{A_j}$	104·5	104·5	104·5	104·5	104·5	104·5	104·5	104·5	($A_j=4{\cdot}608$ in.^{2})	(41)
	(42)	λ_j	0·0400	0·0370	0·0345	0·0323	0·0303	0·0286	0·0270	0·0256		(42)
	(43)	$(41)\times 2\,(1-\lambda_j)$	200·6	201·3	201·8	202·3	202·7	203·0	203·4	203·7		(43)
	(44)	$(41)\times 2\,(1+\lambda_j)$	217·4	216·7	216·2	215·7	215·3	215·0	214·6	214·3		(44)
	(45)	"Diagonal sum" of (43) & (44)		418·7	418·5	418·5	418·4	418·3	418·4	418·3		(45)

	ROW NO. \ j	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*Multiplier*	$n=3$	*5*	*7*	*9*	*11*
a_{nj}	(18)	2·1043	2·1557	3·2511	7·0142	10·2352	13·6664	16·7951	$10^6/[n^4(n^2-1)^2]$	192·90	2·78	0·18	0·02	0·00
	(22) $\times -1$	−0·2919	−0·2827	−0·3973	−0·8149	−1·1342	−1·4420	−1·6920	$10^6/[n^2(n^2-1)^2]$	1736·1	69·4	8·8	1·9	0·6
	(26)	0·0278	0·0257	0·0323	0·0589	0·0764	0·09111	0·1017	$10^6/(n^2-1)^2$	15,625	1736·1	434·0	156·3	69·4
	(40)	2,673	2,879	3,085	3,291	3,496	3,701	3,907	$1/n^2$	0·11111	0·04	0·02041	0·01235	0·00826
	(45)	418·7	418·5	418·5	418·4	418·3	418·4	418·3	1	1	1	1	1	1
b_{nj}	(29) $\times -1$	−1·5229	−1·5112	−3·2984	−5·8554	−7·8879	−10·4495		$10^6/[n^4(n^2-1)^2]$					
	(32)	0·2064	0·1915	0·3922	0·6659	0·8532	1·0773		$10^6/[n^2(n^2-1)^2]$					
	(35) $\times -1$	−0·0190	−0·0168	−0·0298	−0·0465	−0·0557	−0·0659		$10^6/(n^2-1)^2$			AS ABOVE		
	(39) $\times -1$	−1,388	−1,491	−1,594	−1,697	−1,799	−1,902		$1/n^2$					
	(41)	104·5	104·5	104·5	104·5	104·5	104·5		1					
c_{nj}	(36)	0·3136	0·4457	1·2449	1·6940	2·2633	ITEMS IN THESE ROWS ARE DISPLACED ONE COLUMN TO THE LEFT, SINCE c_{nj} COMES FROM THE RIB $j+1$		$10^6/[n^4(n^2-1)^2]$					
	(37) $\times -1$	−0·0409	−0·0552	−0·1456	−0·1881	−0·2393			$10^6/[n^2(n^2-1)^2]$			FIRST THREE ROWS ABOVE		
	(38)	0·0038	0·0046	0·0105	0·0127	0·0151			$10^6/(n^2-1)^2$					

EXAMPLE: $a_{31}=(2{\cdot}1043)\,(192{\cdot}90)-(0{\cdot}2919)\,(1736{\cdot}1)+(0{\cdot}0278)\,(15{,}625)+(2{,}673)\,(0{\cdot}11111)+(418{\cdot}7)\,(1)=1049{\cdot}2$.

TABLE II.

THE LOADING COEFFICIENTS l_{nj}.

	Row No.	Item or operation	$j=0$	1	2	3	4	5	6	7	8	Row No.
TANGENTIAL LOADS	(101)	$2W/\pi$, case G, $\gamma=\pi/2$			11,671						−5,305	(101)
	(102)	(101) × (8) × 10^{-6}			1·4122						−1·2589	(102)
	(103)	(102) × (1) × (6)			1586·6						−2971·0	(103)
	(104)	(102) × {(5) (6) − (11)} × −1			−70·17						111·74	(104)
	(105)	(102) × (9) × (9)			8·826						−7·868	(105)
	(106)	(103) × (3) × −1			−2148·6						5880·2	(106)
	(107)	(103) × {(3) + (4)}			4627·6							(107)
	(108)	(103) × (4) × −1			−2479·0							(108)
	(109)	"Diagonal Sum" of (106), (107) & (108)		−2148·6	4627·6	−2479·0				5880·2		(109)
	(110)	(104) × (3) × −1			95·02						−221·16	(110)
	(111)	(104) × {(3) + (4)}			−204·66							(111)
	(112)	(104) × (4) × −1			109·64							(112)
	(113)	"Diagonal Sum" of (110), (111) & (112)		95·02	−204·66	109·64				−221·16		(113)
	(114)	(105) × (3) × −1			−11·952						15·572	(114)
	(115)	(105) × {(3) + (4)}			25·743							(115)
	(116)	(105) × (4) × −1			−13·791							(116)
	(117)	"Diagonal Sum" of (114), (115) & (116)		−11·952	25·743	−13·791				15·572		(117)
RADIAL LOADS	(118)	$2W/\pi$, case C	12,732			−6,366						(118)
	(119)	(118) × (8) × 10^{-6}	0·7066			−0·8301						(119)
	(120)	(119) × (1) × (6) × −1	−561·7			1077·1						(120)
	(121)	(119) × {(5) (6) − (9) (9) − (11)}	14·026			−42·078						(121)
	(122)	(120) × (3) × −1				−1570·7						(122)
	(123)	(120) × {(3) + (4)}				3365·9						(123)
	(124)	(120) × (4) × −1	760·7			−1795·2						(124)
	(125)	"Diagonal Sum" of (122), (123) & (124)		760·7	−1570·7	3365·9	−1795·2					(125)
	(126)	(121) × (3) × −1				61·362						(126)
	(127)	(121) × {(3) + (4)}				−131·493						(127)
	(128)	(121) × (4) × −1	−18·994			70·131						(128)
	(129)	"Diagonal Sum" of (126), (127) & (128)		−18·994	61·362	−131·493	70·131					(129)

(NOTE: Row numbers less than 100 refer to TABLE I.)

TABLE II (*Continued*)

	Row No.	$j=1$	2	3	4	5	6	7	Multiplier	$n=3$	5	7	9	11
	(109)	−2148·6	4627·6	−2479·0				5880·2	$\frac{10^6 \sin \frac{n\pi}{2}}{n^3(n^2-1)^3}$	−578·70	13·889	−1·265	0·214	−0·052
	(113)	95·02	−204·66	109·64				−221·16	$\frac{10^6 \sin \frac{n\pi}{2}}{n(n^2-1)^3}$	−5208·3	347·22	−62·00	17·36	−6·31
l_{nj}	(117)	−11·952	25·743	−13·791				15·572	$\frac{10^6 n \sin \frac{n\pi}{2}}{(n^2-1)^3}$	−46,875	8680·6	−3038·2	1406·3	−763·9
	(125)	760·7	−1570·7	3365·9	−1795·2				$\frac{10^6 \sin^2 \frac{n\pi}{2}}{n^2(n^2-1)^3}$	1736·1	69·444	8·858	1·929	0·574
	(129)	−18·994	61·362	−131·493	70·131				$\frac{10^6 \sin^2 \frac{n\pi}{2}}{(n^2-1)^3}$	15,625	1736·1	434·03	156·25	69·44

EXAMPLE: $l_{13} = (-2148{\cdot}6)(-578{\cdot}70) + (95{\cdot}02)(-5208{\cdot}3) + (-11{\cdot}952)(-46{,}875) + (760{\cdot}7)(1736{\cdot}1) + (-18{\cdot}994)(15{,}625) = 2{,}333 \times 10^3$

matter of routine. The scheme is set out in Table I, the numerical quantities tabulated being for the example illustrated in Fig. 4, and it deals with the general case where the offset of frame neutral axis (h_j), strain energy of frame end load, and shear strain energy of the frames, are all taken into account—in addition, of course, to the bending strain energy of the frames, the shear strain energy of the skin and the strain energy of the stringers. The scheme is (it is hoped) more or less fool-proof, and only a few words of explanation appear to be necessary. The chief point to note is that each coefficient (a_{nj}, b_{nj}, c_{nj}) is the sum of as many as five products, each of which may be resolved into two factors, one factor being dependent only on the structural data and the other being a function of n; one of these functions is, incidentally, a constant, which is given the value unity.

The upper part of Table I is concerned with the evaluation of the structural factors, which are then tabulated in the lower part of the Table, to the left of the block of "multipliers." The latter are the functions of n referred to above, their numerical values being given for $n=3$, 5, 7, 9 and 11. (The nature of the applied loading is such that $\psi_{nj}=0$ for even values of n, but when the applied loading is unsymmetrically disposed with respect to the plane YOz the tabulation must include both odd and even values of n.)

The lower part of Table I is essentially a convenient device for the "mass-production" of the required coefficients. The computation of the individual coefficients will be clear from the example

$$a_{31}\,(n=3, j=1)$$
$$= (2{\cdot}1043)(192{\cdot}9) - (0{\cdot}2919)(1736{\cdot}1) + (0{\cdot}0278)(15625) + (2673)(0{\cdot}11111) + (418{\cdot}7)(1) = 1049{\cdot}2.$$

The use of a calculating machine with automatic multiplication and accumulation of binary products may be regarded as essential in all this work. With its aid, problems which would otherwise probably never be attempted may be solved in quite a reasonable time, particularly after practice has given the user confidence in the accuracy of his work. The calculation of the coefficient a_{31} given above will be recognised by the reader familiar with matrix or vector theory as the formation of a scalar product. This is just the sort of work which the calculating machine revels in, and it is not being used to the best advantage if it is not given plenty of this to do.

10.2. THE LOADING COEFFICIENTS l_{nj}

The quantity l_{nj} is the term in strain energy equation nj due to the applied loading, and it derives from frames $j-1$, j and $j+1$, in the general case when all three frames are loaded. Thus, referring to equation (46),

l_{nj} = bottom loading term for rib $j-1$ + middle loading term for rib j + top loading term for rib $j+1$.

In the present case, since the given tangential loads are applied on the neutral axes of the frames, the factor (47) is used in place of that given in (46).

The calculation of the coefficients l_{nj} may, as in the case of the structural coefficients a_{nj}, b_{nj} and c_{nj}, be very

much simplified—one might almost say made possible—by appropriate tabulation of the various quantities involved, and the scheme shown in Table II has been designed with this end in view. The tabulation is based on the consideration that l_{nj} may be expressed as a scalar product of a set of quantities depending on the applied loads, with a set of "multipliers," the latter being functions of n depending partly on the type of frame loading, as evidenced by the presence of the factors $\sin n\pi/2$ and $\sin^2 n\pi/2$.

The quantities depending on the applied loads are computed in the upper part of Table II. These are then transferred to the left-hand side of the lower part of the Table, and, with the block of "multipliers" on the right-hand side, the "mass-production" of the loading coefficients may now be undertaken.

It should be noted that the particular arrangement of Table II has been devised for the two types of applied loading (case G with $\gamma=\pi/2$, and case C) to which the structure illustrated in Fig. 4 is subjected, but a similar scheme appropriate to any other type of loading should present no particular difficulty. An important feature of this table, which the given system of applied loads does not fully emphasise, is that it is particularly suitable when there are a larger number of loaded frames, as for example when three or more consecutive frames are subjected to tangential point-loads. The device of "diagonal summation" then comes into action, as shown by the arrows in rows 106, 107 and 108, the "diagonal sum" for columns $j-1$, j and $j+1$ being entered in column j.

10.3. THE SOLUTION OF THE STRAIN ENERGY EQUATIONS

As is shown in Section 9, a set of equations ($j=1$ to J) is obtained for each value of n, of the form given in (57). This makes it possible to solve for the unknowns $\psi_{nj}\,(n>1)$ for each value of n separately, and is a very important feature of the method, since each set of equations is of a simple character readily admitting of speedy solution.

The method adopted for the solution of the equations is due to Choleski, and the layout is exhibited in the left-hand part of Table III. An account of Choleski's method is presented in Ref. 3, but for the sake of completeness a brief description of the method will be given here. The proof is most economically set out in matrix notation.

Let the system of equations be written in the form

$$A\psi+L=0,$$

where A is the symmetrical matrix of structural coefficients, and L the column of loading terms.

Put $A=TT'$, where T is a lower triangular matrix, and T' is its transpose. As is seen below, the resolution of A into a product of this form can in fact be made, and the details of the process make it evident that the result is unique.

Thus, $TT'\psi+L=0,$

and writing $T'\psi+Y=0 \quad . \quad . \quad (58)$

then $TY=L \quad . \quad . \quad . \quad (59)$

$e\ \downarrow$	A_{11},	A_{12},	A_{13},	A_{14},	L_1	ψ
e_1	u_{11}	A_{22},	A_{23},	A_{24},	L_2	ψ_1
e_2	u_{12}	u_{22}	A_{33},	A_{34},	L_3	ψ_2
e_3	u_{13}	u_{23}	u_{33}	A_{44},	L_4	ψ_3
e_4	u_{14}	u_{24}	u_{34}	u_{44}		ψ_4
$Y'\rightarrow$	y_1	y_2	y_3	y_4		
$s\rightarrow$	s_1	s_2	s_3	s_4		

FIGURE 5.

Y is a column of subsidiary variables defined by (59), the original variables (ψ) being derived therefrom by the solution of (58). Both (58) and (59) are triangular sets of equations which may be solved by direct substitution, starting from the "tip" of the triangle in each case, and the matrix of coefficients for one set is simply the transpose of the matrix for the other set. The solution of the original equations is thus made to depend on the resolution of the matrix A into the product TT', *i.e.* on the determination of the matrix T.

The coefficients of T and the terms in Y may be found by certain easily remembered rules whose derivation is most simply illustrated by taking a moderately simple example. For this purpose, assume that there are only four unknowns, for which

$$T=\begin{bmatrix} u_{11} & . & . & . \\ u_{12}, & u_{22} & . & . \\ u_{13}, & u_{23}, & u_{33} & . \\ u_{14}, & u_{24}, & u_{34}, & u_{44} \end{bmatrix} \text{ and } Y=\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$$\text{Also, let } A=\begin{bmatrix} A_{11}, & A_{12}, & A_{13}, & A_{14} \\ * & A_{22}, & A_{23}, & A_{24} \\ * & * & A_{33}, & A_{34} \\ * & * & * & A_{44} \end{bmatrix} \text{ and } L=\begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{bmatrix}$$

The asterisks in A signify a symmetrical matrix, and represent terms (the transpose of those above the leading diagonal) which in practice need not actually be written down. Then

$$A_{11}=u_{11}^2,\ A_{12}=u_{11}u_{12},\ A_{13}=u_{11}u_{13},\ A_{14}=u_{11}u_{14}$$

and $L_1=u_{11}y_1$, giving

$$u_{11}=(A_{11})^{\frac{1}{2}},\ u_{12}=\frac{A_{12}}{u_{11}},\ \text{etc.,}\ \ldots\ y_1=\frac{L_1}{u_{11}} \quad . \quad (60)$$

Next, $A_{22}=u_{12}^2+u_{22}^2,\ A_{23}=u_{12}u_{13}+u_{22}u_{23},$

$A_{24}=u_{12}u_{14}+u_{22}u_{24}$ and $L_2=u_{12}y_1+u_{22}y_2,$

whence

$$u_{22}=(A_{22}-u_{12}^2)^{\frac{1}{2}},$$

$$u_{23}=\frac{1}{u_{22}}(A_{23}-u_{12}u_{13}),\ \text{etc.,}\ y_2=\frac{1}{u_{22}}(L_2-u_{12}y_1) \quad (61)$$

$e \downarrow$	$a_{n,1}$	$b_{n,1}$	$c_{n,1}$	.	.	.	.	$l_{n,1}$	ψ
e_1	u_{11}	$a_{n,2}$	$b_{n,2}$	$c_{n,2}$	.	.	.	$l_{n,2}$	$\psi_{n,1}$
e_2	u_{12}	u_{22}	$a_{n,3}$	$b_{n,3}$	$c_{n,3}$	.	.	$l_{n,3}$	$\psi_{n,2}$
e_3	u_{13}	u_{23}	u_{33}	$a_{n,4}$	$b_{n,4}$	$c_{n,4}$	.	$l_{n,4}$	$\psi_{n,3}$
e_4	.	u_{24}	u_{34}	u_{44}	$a_{n,5}$	$b_{n,5}$	$c_{n,5}$	$l_{n,5}$	$\psi_{n,4}$
e_5	.	.	u_{35}	u_{45}	u_{55}	$a_{n,6}$	$b_{n,6}$	$l_{n,6}$	$\psi_{n,5}$
e_6	.	.	.	u_{46}	u_{56}	u_{66}	$a_{n,7}$	$l_{n,7}$	$\psi_{n,6}$
e_7	.	.	.	.	u_{57}	u_{67}	u_{77}		$\psi_{n,7}$
$Y' \rightarrow$	y_1	y_2	y_3	y_4	y_5	y_6	y_7		
$s \rightarrow$	s_1	s_2	s_3	s_4	s_5	s_6	s_7		

FIGURE 6.

Further,

$$A_{33}=u_{13}^2+u_{23}^2+u_{33}^2, A_{34}=u_{13}u_{14}+u_{23}u_{24}+u_{33}u_{34}$$

and $$L_3=u_{13}y_1+u_{23}y_2+u_{33}y_3,$$

so that
$$\left.\begin{aligned} u_{33}&=(A_{33}-u_{23}^2-u_{13}^2)^{\frac{1}{2}},\\ u_{34}&=\frac{1}{u_{33}}(A_{31}-u_{13}u_{14}-u_{23}u_{24})\\ \text{and}\quad y_3&=\frac{1}{u_{33}}(L_3-u_{13}y_1-u_{23}y_2)\end{aligned}\right\} \quad . \quad . \quad (62)$$

Finally, $A_{44}=u_{14}^2+u_{24}^2+u_{34}^2+u_{44}^2$

and $$L_4=u_{14}y_1+u_{24}y_2+u_{34}y_3+u_{44}y_4,$$

giving $$u_{44}=(A_{44}-u_{14}^2-u_{24}^2-u_{34}^2)^{\frac{1}{2}}$$

and $$y_4=\frac{1}{u_{44}}(L_4-u_{14}y_1-u_{24}y_2-u_{34}y_3) \quad . \quad (63)$$

A very economical lay-out is obtained if the coefficients of T and Y are written below the leading diagonal of A, separating them from those of the latter by a thick zig-zag line, the loading column (L) being put at the right-hand side of A. This is shown in Fig. 5, which also includes a column e and a row s whose purpose will shortly be explained, and a column ψ for the final solution.

From (60) we see that the column below A_{11} (not including s_1) is the first row of the matrix $[A, L]$, divided by $(A_{11})^{\frac{1}{2}}$.

The column below A_{22} then follows from equation (61), the column below A_{33} from equation (62), and so on. The extension to any number of variables will be obvious.

With the modern electrically-operated desk calculator the computations are, with practice, rapidly done, since all the intermediate operations for any particular coefficient are carried in the machine, and only the final answer need be written down.

The column e and the row s are used for checking purposes, the terms in s being the sums of the items in the corresponding columns (up to the zig-zag line).

Thus, $$u_{11}s_1=u_{11}(u_{11}+\ldots+u_{14}+y_1)$$
$$=A_{11}+A_{12}+\ldots+A_{14}+L_1, \text{ from (60)}$$

or $$A_{11}+A_{12}+\ldots+A_{14}+L_1-u_{11}s_1=0 \quad . \quad (64)$$

Similarly, from (60) and (61)

$$u_{12}s_1+u_{22}s_2=u_{12}(u_{11}+u_{12}+\ldots+u_{14}+y_1)+$$
$$+u_{22}(u_{22}+u_{23}+\ldots+y_2)$$
$$=A_{12}+A_{22}+A_{23}+A_{24}+L_2,$$

or $$A_{12}+A_{22}+A_{23}+A_{24}+L_2-(u_{12}s_1+u_{22}s_2)=0$$
$$. \quad . \quad . \quad (65)$$

Proceeding in this fashion, from (60), (61) and (62)

$$u_{13}s_1+u_{23}s_2+u_{33}s_3=u_{13}(u_{11}+\ldots+y_1)+$$
$$+u_{23}(u_{22}+\ldots+y_2)+u_{33}(u_{33}+\ldots+y_3)$$
$$=A_{13}+A_{23}+A_{33}+A_{34}+L_3,$$

or $$A_{13}+A_{23}+A_{33}+A_{34}+L_3-$$
$$-(u_{13}s_1+u_{23}s_2+u_{33}s_3)=0, \quad . \quad . \quad (66)$$

and so on.

The residual values of the left-hand sides of (64), (65) and (66), etc. are entered in column e as e_1, e_2, e_3, etc., as each successive check is carried out. These entries are really an expression of satisfaction on the part of the computor at the completion of his labours on each column, since the check will not succeed if there are any errors in the column, and until such errors are eliminated the entry in column e cannot be made.

Having completed the matrix T and the row Y', the original variables ψ_1, ψ_2, ψ_3 and ψ_4 now follow from equation (58). Thus, referring again to Fig. 5,

$$u_{44}\psi_4+y_4=0, \text{ whence } \psi_{44}=-y_4/u_{44},$$

next $$u_{33}\psi_3+u_{34}\psi_4+y_3=0,$$

giving ψ_3, and so on.

An overall check on the ψ's may be obtained by summing each row of T and forming the scalar product of these sums with the column of ψ's. The result should be equal and opposite to Σy, since

$$u_{11}\psi_1+(u_{12}+u_{22})\psi_2+(u_{13}+u_{23}+u_{33})\psi_3+$$
$$+(u_{14}+u_{24}+u_{34}+u_{44})\psi_4+(y_1+y_2+y_3+y_4)=0.$$

Alternatively, a similar check may be made using the coefficients of the original equations; in this case it is verified that

$$(A_{11}+A_{12}+A_{13}+A_{14})\psi_1+(A_{12}+A_{22}+A_{23}+A_{24})\psi_2+$$
$$+(A_{13}+A_{23}+A_{33}+A_{34})\psi_3+(A_{14}+A_{24}+A_{34}+A_{44})\psi_4+$$
$$+(L_1+L_2+L_3+L_4)=0.$$

The lay-out—corresponding to Fig. 5—for a typical set of strain energy equations (57) is shown in Fig. 6. This should be compared with the left-hand part of Table III, where five sets of equations have been solved according to this plan.

TABLE III.

SOLUTION OF THE STRAIN ENERGY EQUATIONS, ETC.

n	e \ j	1	2	3	4	5	6	7	$L \times 10^{-3}$	$\psi_{nj} \times 10^{-3}$	nj
3		1049·2	−282·0	48·9	·	·	·		2333·2		
	0·020	32·391	1065·0	−282·7	62·0	·	·	·	−4587·9	−1·2487	31
	−0·036	−8·706	31·452	1203·3	−493·6	151·4	·	·	5300·5	2·9958	32
	0·026	1·510	−8·570	33·579	1642·7	−784·1	198·6	·	−2021·5	−3·6457	33
	−0·001	·	1·971	−14·197	37·911	2005·8	−1006·0	257·1	·	0·2352	34
	−0·056	·	·	4·509	−18·994	40·308	2385·8	−1281·9	·	0·7288	35
	−0·056	·	·	·	5·239	−22·489	43·042	2743·8	−2980·4	1·1144	36
	−0·020	·	·	·	·	6·378	−26·450	44·761		1·5386	37
		72·032	−125·931	122·472	−0·911	−14·129	−7·271	−68·868			
	s	97·227	−101·078	146·363	23·245	10·068	9·321	−24·107			
5	−0·017	559·4	26·1	4·6	·	·	·	·	−80·7		
	−0·014	23·652	564·7	24·8	5·4	·	·	·	214·2	0·1610	51
	0·012	1·104	23·738	579·5	7·0	11·6	·	·	−110·6	−0·3959	52
	0·017	0·194	1·036	24·050	615·4	−14·3	13·7	·	−2·9	0·2064	53
	0·004	·	0·227	0·281	24·805	640·8	−27·0	15·9	·	0·0062	54
	0·012	·	·	0·282	−0·582	25·303	662·7	−40·5	·	0·0010	55
	−0·010	·	·	·	0·552	−1·054	25·715	680·7	140·0	−0·0127	56
		·	·	·	·	0·628	−1·549	26·037		−0·2064	57
		−3·412	9·182	−4·967	−0·145	0·091	0·007	5·375			
	s	21·538	34·183	19·846	24·630	24·968	24·173	31·412			
7		483·1	69·4	1·3	·	·	·	·	31·6		
	−0·015	21·980	486·4	68·2	1·6	·	·	·	−58·7	−0·0852	71
	0·025	3·157	21·827	492·6	61·8	3·5	·	·	11·0	0·1382	72
	0·006	0·059	3·116	21·975	505·3	54·4	4·2	·	14·5	−0·0381	73
	0·028	·	0·073	2·802	22·303	514·9	49·6	4·9	·	−0·0247	74
	−0·000	·	·	0·159	2·419	22·562	523·4	44·5	·	0·0028	75
	0·024	·	·	·	0·188	2·178	22·773	530·5	−41·0	−0·0067	76
	0·024	·	·	·	·	0·217	1·933	22·950		0·0778	77
		1·438	−2·897	0·907	0·546	−0·065	0·002	−1·786			
	s	26·634	22·119	25·843	25·456	24·892	24·708	21·164			
9		455·5	84·8	0·5	·	·	·	·	−17·1		
	0·031	21·342	457·6	83·8	0·6	·	·	·	40·2	0·0591	91
	0·005	3·973	21·019	460·9	80·8	1·4	·	·	−32·1	−0·1165	92
	−0·016	0·023	3·983	21·096	466·9	77·4	1·7	·	7·5	0·0966	93
	−0·037	·	0·029	3·825	21·267	471·5	75·0	2·0	·	−0·0334	94
	−0·003	·	·	0·066	3·628	21·409	475·8	72·5	·	0·0045	95
	−0·031	·	·	·	0·080	3·490	21·532	479·6	19·3	0·0057	96
	−0·016	·	·	·	·	0·093	3·352	21·642		−0·0411	97
		−0·801	2·064	−1·910	0·693	−0·112	0·016	0·890			
	s	24·537	27·095	23·077	25·668	24·880	24·900	22·532			
11		442·5	91·8	0·2	·	·	·	·	7·8		
	−0·029	21·036	443·9	91·1	0·3	·	·	·	−15·3	−0·0265	111
	−0·013	4·364	20·612	446·0	89·5	0·6	·	·	2·8	0·0428	112
	−0·004	0·010	4·418	20·651	449·2	87·7	0·8	·	3·8	−0·0138	113
	−0·022	·	0·015	4·331	20·747	451·8	86·3	0·9	·	−0·0062	114
	0·021	·	·	0·029	4·221	20·832	454·4	84·9	·	0·0021	115
	−0·024	·	·	·	0·039	4·135	20·912	456·6	−10·8	−0·0050	116
	0·004	·	·	·	·	0·043	4·051	20·981		0·0246	117
		0·371	−0·821	0·311	0·119	−0·025	0·005	−0·516			
	s	25·781	24·224	25·322	25·126	24·985	24·968	20·465			

TABLE III (*Continued*)

	j \ n	1	3	5	7	9	11
	0	0	0	0	0	0	0
ψ_{nj}	1	−9402	−1249	161	−85	59	−27
(lb.)	2	−17462	2996	−396	138	−117	43
(*See*	3	−16977	−3646	206	−38	97	−14
Fig. 4	4	−12732	235	6	−25	−33	−6
for	5	−8988	729	1	3	5	2
$\psi_{1,j}$)	6	−5659	1114	−13	−7	6	−5
	7	−2681	1539	−206	78	−41	25
	8	0	0	0	0	0	0
	0	9402	416	−32	12	−7	2
	1	8060	−1415	111	−32	20	−6
$\frac{1}{n}(\psi_{nj}-$	2	−485	2214	−120	25	−24	5
$-\psi_{n,j+1})$	3	−4245	−1294	40	−2	14	−1
$\frac{}{n}$	4	−3744	−165	1	−4	−4	−1
(lb.)	5	−3329	−128	3	1	0	1
	6	−2978	−142	39	−12	5	−3
	7	−2681	513	−41	11	−5	2
	0		211	−9	2	−1	0
$\frac{nP_{nj}}{n^2-1}$	1		−969	42	−9	4	−1
	2		2016	−71	12	−7	2
(lb.)	3		−2019	50	−6	7	−1
See	4		649	−13	−1	−3	0
equa-	5		13	1	1	1	0
tion	6		−20	15	−4	1	−1
(38)	7		500	−33	7	−2	1
	8		−381	17	−3	1	0

	$N_j R_j \cos\alpha$ (lb.)			$\frac{(qR^2)_j L_j}{R_j R_{j+1}}$ (lb.)		
$\theta°$ \ j	2	3	7	1	2	3
0	−14798	−20372	−1286	0	0	0
5	−14805	−20218	−1322	374	482	−680
10	−14824	−19752	−1422	761	935	−1342
15	−14846	−18969	−1570	1174	1333	−1967
20	−14864	−17869	−1752	1624	1646	−2534
25	−14874	−16480	−1960	2120	1849	−3020
30	−14862	−14860	−2189	2670	1921	−3409
35	−14787	−13096	−2429	3278	1846	−3690
40	−14588	−11288	−2653	3943	1627	−3862
45	−14202	−9521	−2830	4653	1276	−3934
50	−13590	−7849	−2933	5389	819	−3922
55	−12755	−6295	−2953	6132	284	−3842
60	−11718	−4863	−2890	6862	−299	−3712
65	−10485	−3564	−2741	7562	−902	−3547
70	−9018	−2431	−2487	8213	−1495	−3365
75	−7248	−1508	−2092	8786	−2037	−3186
80	−5121	−821	−1531	9243	−2479	−3034
85	−2662	−347	−813	9540	−2771	−2931
90	0	0	0	9644	−2873	−2894

11. Numerical Results for the Example of Figure 4

The derivation of the unknowns (ψ_{nj}) in the strain energy equations is an essential step towards the complete solution of the problem, but the shear and end load distributions in the skin and stringers, and the frame bending moments, end loads and shears have still to be calculated.

In view of the large amount of information which can now be obtained, a systematic tabulation of the required quantities is almost a necessity on the grounds of speed and accuracy. A few notes on this subject, illustrated by the example of Fig. 4, will therefore not be out of place here.

First note that, in Table III, the loading column L has been divided by 1,000. This is done to reduce the loading terms to the same order of magnitude as the structural coefficients, the purpose being to keep the number of significant figures needed for the checking process down to a minimum. The column on the right of $L \times 10^{-3}$ then gives $\psi_{nj} \times 10^{-3}$.

The full values of ψ_{nj} (rounded off to the nearest whole number) are then tabulated in the upper block on the right-hand side of Table III, and below these are entered the quantities $(\psi_{nj} - \psi_{n,j+1})/n$. These latter are the coefficients in the sine series for $qR_2L_j/(R_j R_{j+1})$ (*see* equation (8)), while in the present example $\phi_{nj} = \tau_j = 0$.

The third block on the right-hand side of Table III gives $nP_{nj}/(n^2-1)$, P_{nj} being derived from equation (38), and is required for the calculation of frame bending moments (31), end loads (33) and shears (35).

The shear and end load distributions in the skin and stringers may now be computed from equations (7) and (8) respectively. For this purpose tables are needed of sines and cosines of multiples of θ (θ, 2θ, etc., up to 11θ or 12θ, say), over the range $\theta = 0$ to 180°, an interval of 5° or even 10°, being quite good enough for practical use. Such tables do not seem to be readily available in print, although they must surely be used extensively in harmonic analysis, but they can very quickly be made up if, as suggested, a fairly large interval of tabulation is acceptable.

The calculation of any particular function, for a range of values of θ, is facilitated by writing the coefficients of the series on a strip of paper so that, when placed upon these home-made trigonometrical tables, the proper factor will appear against each $\cos n\theta$ or $\sin n\theta$, as the case may be. The sum of products is then obtained on the calculating machine and the answer tabulated or plotted as desired.

In the present example the most interesting distributions are $N_j R_j \cos\alpha$ at frames 2, 3 and 7, and $(qR^2)_j L_j/(R_j R_{j+1})$ in bays 1, 2 and 3, and these are therefore put in Table III, below $nP_{nj}/(n^2-1)$. The complete set of distributions was afterwards computed, and plotted as Figs. 7 and 8. The first of these diagrams shows $N_j R_j \cos\alpha$ divided by $\psi_{1,j}$, while the second diagram shows $(qR^2)_j L_j/(R_j R_{j+1})$ divided by $(\psi_{1,j} - \psi_{1,j+1})$. The "elementary theory" values of these functions are respectively $\cos\theta$ and $\sin\theta$, which are accordingly included for comparison.

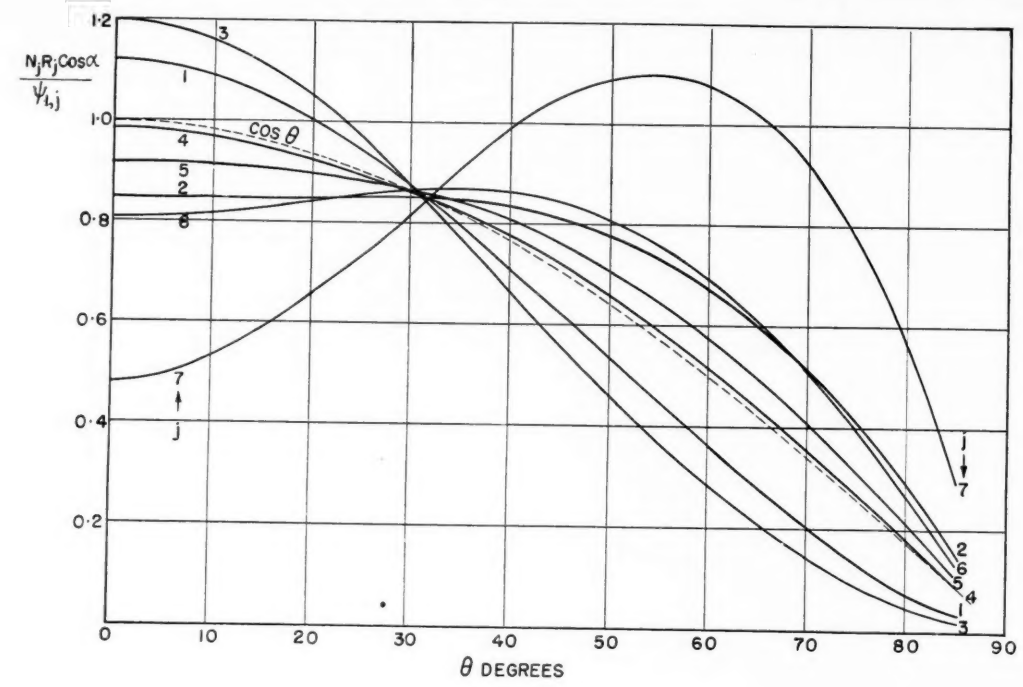


FIGURE 7.

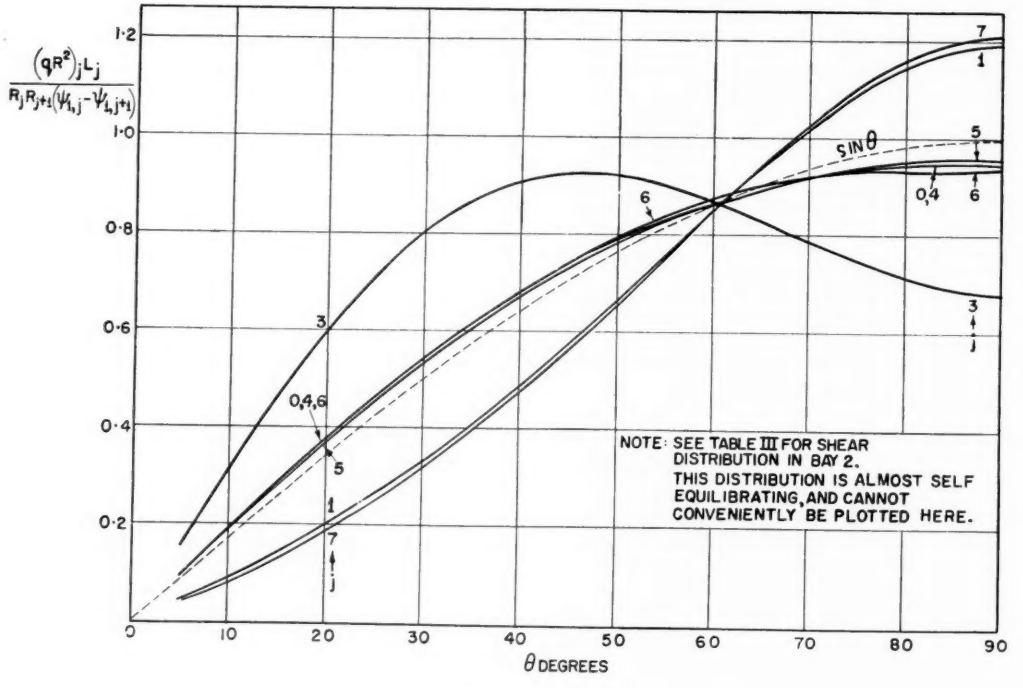


FIGURE 8.

FRAME BENDING MOMENTS, END LOADS, AND SHEARS

The most convenient formulae for numerical work are (31), (33) and (35), provided simple expressions can be obtained for $\overline{M}$, $\overline{T}$ and $\overline{S}$. For radial or tangential point loads these are more or less standard results, but it is interesting to derive them from equations (29), (32) and (34), by summation of the trigonometrical series. Only the cases required for the analysis of the frames in the numerical example will be considered.

11.1. RADIAL LOADS SYMMETRICAL ABOUT THE PLANE XOz

The general expressions for $\overline{M}$, $\overline{T}$ and $\overline{S}$ for loading symmetrical about the plane XOz (the "vertical" plane in Fig. 2) are obtained from (29), (32) and (34), respectively, by retaining only the coefficients $\overline{K}_{nj}$ in these equations. Thus

$$\left.\begin{aligned} \overline{M} &= (R_j - h_j)\sum_{n=2}\left\{\frac{\overline{K}_{nj}\cos n\theta}{(n^2-1)}\right\} - \frac{\rho_j{}^2\overline{K}_{0,j}}{(R_j-h_j)}, \\ \overline{T} &= \overline{K}_{0,j} + \frac{R_j K_{1,j}\cos\theta}{(R_j-h_j)} - \sum_{n=2}\left\{\frac{\overline{K}_{nj}\cos n\theta}{(n^2-1)}\right\}. \\ \text{and } \overline{S} &= \frac{h_j\overline{K}_{1,j}\sin\theta}{(R_j-h_j)} - \sum_{n=2}\left\{\frac{n\overline{K}_{nj}\sin n\theta}{(n^2-1)}\right\} \end{aligned}\right] \quad (67)$$

The cases of interest here are A and C; for case A, $\overline{K}_{0,j} = W/2\pi$ and $\overline{K}_{nj}(n>0) = W/\pi$, and hence the sum $\sum_{n=2}\{\cos n\theta/(n^2-1)\}$ is required. Differentiation with respect to θ will then give the sum $\sum_{n=2}\{-n\sin n\theta/(n^2-1)\}$. (n^2-1).

Now $\sum_{n=2}\{\cos n\theta/(n^2-1)\}$ may be shown to be the Fourier series for

$$\{\tfrac{1}{2} + \tfrac{1}{4}\cos\theta - \tfrac{1}{2}(\pi-\theta)\sin\theta\}$$

in the range $0 \leqslant \theta \leqslant 2\pi$,

whence

$$\sum_{n=2}\left(\frac{-n\sin n\theta}{n^2-1}\right) = \frac{1}{4}\sin\theta - \frac{1}{2}(\pi-\theta)\cos\theta \text{ for } 0<\theta<2\pi.$$

The term $-\rho_j{}^2K_{0,j}/(R_j-h_j)$ will be neglected, as being insignificant in practice.

It is then found that, for case A,

$$\left.\begin{aligned} \overline{M} &= \frac{W(R_j-h_j)}{\pi}\left\{\frac{1}{2} + \frac{1}{4}\cos\theta - \frac{1}{2}(\pi-\theta)\sin\theta\right\}, \\ \overline{T} &= \frac{W}{\pi}\left\{\left(\frac{R_j}{R_j-h_j} - \frac{1}{4}\right)\cos\theta + \frac{1}{2}(\pi-\theta)\sin\theta\right\}, \\ \text{and } \overline{S} &= \frac{W}{\pi}\left\{\left(\frac{h_j}{R_j-h_j} + \frac{1}{4}\right)\sin\theta - \frac{1}{2}(\pi-\theta)\cos\theta\right\}, \end{aligned}\right] \quad (68)$$

taking θ in the range from 0 to 2π.

Note the discontinuity in $\overline{S}$ at the two ends of the range, due to the applied load.

For case C, $\overline{K}_{0,j} = 0$ and $\overline{K}_{nj} = 2W/\pi$ (n odd). Thus the sum $\sum_{n=3}\{\cos n\theta/(n^2-1)\}$ is required (odd terms only).

Now this latter series may be expressed in the form

$$\frac{1}{2}\sum_{n=2}\left\{\frac{\cos n\theta - \cos n(\pi-\theta)}{n^2-1}\right\}$$

and it is hence the Fourier series for the function

$$\tfrac{1}{2}\{\tfrac{1}{2} + \tfrac{1}{4}\cos\theta - \tfrac{1}{2}(\pi-\theta)\sin\theta\} - \\ - \tfrac{1}{2}\{\tfrac{1}{2} + \tfrac{1}{4}\cos(\pi-\theta) - \tfrac{1}{2}\theta\sin(\pi-\theta)\},$$

i.e. for $$\left\{\frac{1}{4}\cos\theta + \frac{1}{2}\left(\theta - \frac{\pi}{2}\right)\sin\theta\right\},$$

the range now being $0 \leqslant \theta \leqslant \pi$.

Differentiation with respect to θ then gives

$$\sum_{n=3}\left(\frac{-n\sin n\theta}{n^2-1}\right)(\text{odd terms only}) = \frac{1}{4}\sin\theta + \frac{1}{2}\left(\theta - \frac{\pi}{2}\right)\cos\theta, \quad 0<\theta<\pi.$$

Thus for case C,

$$\left.\begin{aligned} \overline{M} &= \frac{W(R_j-h_j)}{\pi}\left\{\frac{1}{2}\cos\theta + \left(\theta - \frac{\pi}{2}\right)\sin\theta\right\}, \\ \overline{T} &= \frac{W}{\pi}\left\{\left(\frac{2R_j}{R_j-h_j} - \frac{1}{2}\right)\cos\theta - \left(\theta - \frac{\pi}{2}\right)\sin\theta\right\}, \\ \text{and } \overline{S} &= \frac{W}{\pi}\left\{\left(\frac{2h_j}{R_j-h_j} + \frac{1}{2}\right)\sin\theta + \left(\theta - \frac{\pi}{2}\right)\cos\theta\right\}, \end{aligned}\right] \quad (69)$$

taking θ in the range from 0 to π.

11.2. TANGENTIAL LOADS SYMMETRICAL ABOUT THE PLANE XOz

We are here interested in case G, with $\gamma = \pi/2$, giving $\overline{P}_{nj} = 2W\sin(n\pi/2)/\pi$. The applied loads are on the neutral axis of the frame, whence $\overline{G}_{nj} = -\overline{P}_{nj}h_j/R_j$ (*see* derivation of (47)).

Equations (29), (32) and (34) then reduce to

$$\overline{M} = -\frac{2W(R_j-h_j)}{\pi}\sum_{n=3}\frac{\sin\frac{n\pi}{2}\cos n\theta}{n(n^2-1)}$$

$$\overline{T} = \frac{2W}{\pi}\left\{-\frac{h_j\cos\theta}{R_j-h_j} + \sum_{n=3}\frac{n\sin\frac{n\pi}{2}\cos n\theta}{n^2-1}\right\}$$

$$\text{and } \overline{S} = \frac{2W}{\pi}\left\{-\frac{h_j\sin\theta}{R_j-h_j} + \sum_{n=3}\frac{\sin\frac{n\pi}{2}\sin n\theta}{n^2-1}\right\}$$

Now $$\sum_{n=3}\left(\frac{\cos n\theta}{n^2-1}\right)(\text{odd terms only}) = \frac{1}{4}\cos\theta + \\ + \frac{1}{2}\left(\theta - \frac{\pi}{2}\right)\sin\theta, \quad 0 \leqslant \theta \leqslant \pi.$$

Integrate both sides of this equation with respect to θ, between the limits 0 and θ, and then replace θ by $\pi/2 - \theta$. This leads to the result

$$\sum_{n=3}\frac{\sin\frac{n\pi}{2}\cos n\theta}{n(n^2-1)} = \frac{3}{4}\cos\theta - \frac{\pi}{4} + \frac{1}{2}\theta\sin\theta, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$$

Differentiation with respect to θ now gives

$$-\sum_{n=3}\frac{\sin\frac{n\pi}{2}\sin n\theta}{n^2-1}=-\frac{1}{4}\sin\theta+\frac{1}{2}\theta\cos\theta,\quad -\frac{\pi}{2}\leqslant\theta\leqslant\frac{\pi}{2}.$$

A further differentiation will give the series required for $\overline{T}$; this is confirmed by equation (23), which is now $\overline{T}=d\overline{S}/d\theta$.

$$\left.\begin{aligned}\text{Hence}\quad \overline{M}&=\frac{W(R_j-h_j)}{\pi}\left(\frac{\pi}{2}-\theta\sin\theta-\frac{3}{2}\cos\theta\right),\\ \overline{T}&=\frac{W}{\pi}\left\{-\left(\frac{2h_j}{R_j-h_j}+\frac{1}{2}\right)\cos\theta+\theta\sin\theta\right\},\\ \text{and}\quad \overline{S}&=\frac{W}{\pi}\left\{\left(-\frac{2h_j}{R_j-h_j}+\frac{1}{2}\right)\sin\theta-\theta\cos\theta\right\},\end{aligned}\right\}\quad .\quad .\quad .\quad .\quad (70)$$

in the range $-\pi/2<\theta<\pi/2$. At $\theta=\pm\pi/2$, $\overline{T}$ is discontinuous, being equal to $+W/2$ and $-W/2$ on the upper and lower cross sections respectively.

In the range $\pi/2<\theta<3\pi/2$, $\overline{M}$, $\overline{T}$ and $\overline{S}$ are obtainable from equations (70) by reflection about the horizontal diameter ($\theta=\pi/2$), with a reversal of sign in the case of the first two of these quantities.

Returning to the numerical results for the example of Fig. 4, formulæ can now be obtained for the bending moment, end loads, and shear in any of the frames. The unloaded frames—*i.e.* those not subjected to any externally applied loads—are only lightly stressed, as can be seen by reference to the values of $nP_{nj}/(n^2-1)$ in Table III. The simplest criterion to take here is the sum of these coefficients,• which gives the maximum tension in the frame (at $\theta=0$).

Of the loaded frames, the ones which show the greatest departure from elementary theory are frames 2 and 3, and this section is concluded with a brief discussion of the results for frame 3.

For $j=3$,

$$R_j=37{\cdot}5\text{ in.},\quad h_j=2{\cdot}9\text{ in.},\quad \frac{W}{\pi}=-3183\text{ lb.}$$

Equations (69), (31), (33) and (35) then give

$$M=(-3183)(34{\cdot}6)\left\{\frac{1}{2}\cos\theta+\left(\theta-\frac{\pi}{2}\right)\sin\theta\right\}-$$
$$-(1{\cdot}267)(-2019)\cos3\theta+(1{\cdot}4)(50)\cos5\theta,$$

$$T=(-3183)\left\{1{\cdot}6676\cos\theta-\left(\theta-\frac{\pi}{2}\right)\sin\theta\right\}-$$
$$-2019\cos3\theta+50\cos5\theta,$$

and

$$S=(-3183)\left\{0{\cdot}6676\sin\theta+\left(\theta-\frac{\pi}{2}\right)\cos\theta\right\}-$$
$$-\frac{1}{3}(2019)\cos3\theta+\frac{1}{5}(50)\cos5\theta.$$

Terms beyond $n=5$ have been neglected.

The maximum (numerical) value of M occurs at $\theta=0$, and is $-52{,}440$ lb. in. This is 95 per cent. of the corresponding value of $\overline{M}$ ($-55{,}070$ lb. in.)

The value of T at $\theta=0$ is $-7{,}277$ lb., which is $1{\cdot}37\overline{T}$. The maximum (numerical) value of T occurs at about $\theta=7°$, and is approximately $-7{,}800$ lb. This may be compared with the maximum (numerical) value of $\overline{T}$, which occurs at about $\theta=24°$ and is approximately $-6{,}340$ lb.

REFERENCES

1. GOODEY, W. J. (1946). The Stresses in a Circular Fuselage. *Journal of the Royal Aeronautical Society*, November 1946.
2. DALLISON, K. J. (1953). Stress Analysis of Circular Frames in a Non-tapering Fuselage. *Journal of the Royal Aeronautical Society*, March 1953.
3. FOX, L., HUSKEY, H. D. and WILKINSON, J. H. (1948). Notes on the Solution of Algebraic Linear Simultaneous Equations. *Quarterly Journal of Mechanics and Applied Mathematics*, June 1948.

Some Problems Associated with Stress Concentration

by

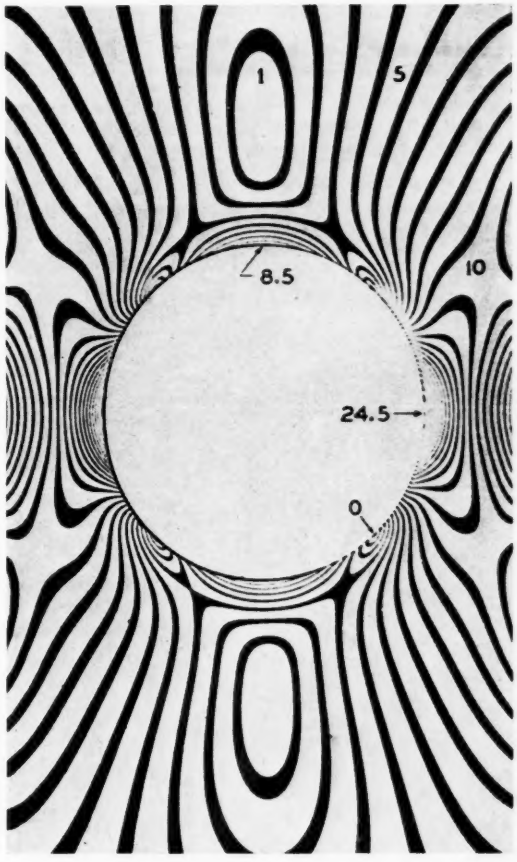
H. L. COX, M.A., F.R.Ae.S.
(Senior Principal Scientific Officer, National Physical Laboratory)

THE theory of stress concentration is such a fascinating study that the whole of this lecture could be most happily devoted to discussion of the stress distribution round all sorts of special boundaries. At the same time the most detailed knowledge of elastic stress distribution is of little value unless we can be sure how the stresses computed will be reflected in the actual strength of the part. For that reason this paper is divided into three sections: the first deals with certain moderately simple cases of stress and load concentration and shows how the stress analysis is confirmed by static and fatigue tests: the second describes some investigations of special two-dimensional boundaries and indicates some fairly general theoretical conclusions which can be drawn: in the third I shall try to review the possible reasons why, in practice, the best found conclusions are not always borne out.

Stress Concentrations

Figure 1 shows a straightforward stress concentration, a hole in a strip. Photoelasticity shows where the maximum stress is developed and its magnitude: moreover from the pattern of fringes the stress distribution can be plotted over the entire field. The concentration of stress arises simply because the stress lines have to flow round the obstacle, the hole; the next case of a rivet transmitting a load to a plate (Fig. 2) is different because the stress lines have in effect to flow into the hole. I usually try to distinguish the two cases, because concentration of load is really a separable effect. For instance, if we cut away a corner of the plate, leaving the rivet supported on one side only, the least possible multiplication of stress at the side of the hole is by about 10, when the depth of section is about twice the rivet diameter; but the stress in a plain beam of the same section under the same loading would be 5·5, so that the real *stress* concentration due to curvature of the beam is only 10/5·5 or about 1·8. Fig. 3 shows both stress and load concentration together: the point again is, of course, that a good part of the high concentration near the point of application of the load is inevitable, whereas the concentration in the fillet is evitable or nearly so.

Whether the concentration is avoidable or not, how does it affect the strength? Fig. 4 shows a standard B.S.I. sling hook, with the maximum stress a little above the horizontal section; Fig. 5 shows this and other hooks after static impact and fatigue tests. Under static and impact tests the hooks just opened out and the stress concentration is evidenced only by the marked extension in these regions; under repeated loading, however, cracks developed, and in the sling hook the crack lay precisely at the point of highest stress. (I may add in parenthesis that we did not expect this result when we put the test in hand; we designed the hook with great care all the way from the bed to the horizontal section, but we designed the shank, between the horizontal section and the eye, by eye. As you see



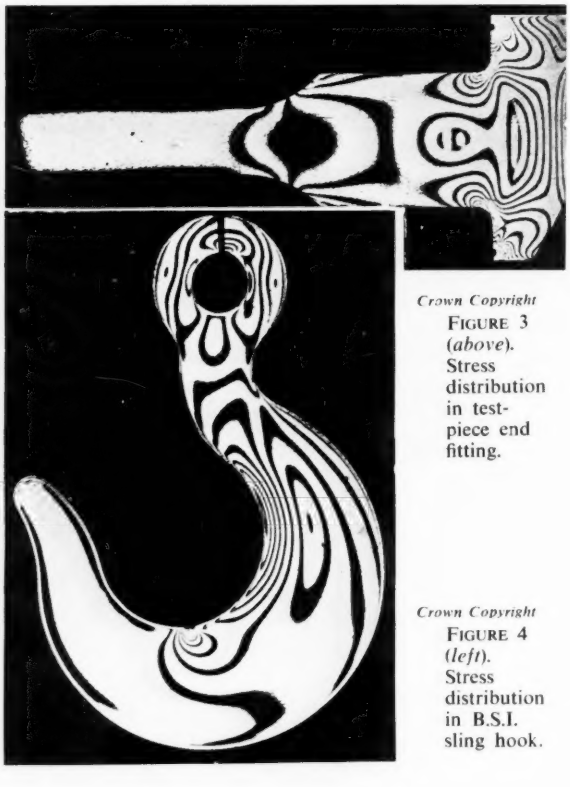
From Frocht's "Photoelasticity" published by John Wiley

FIGURE 1. Stress pattern of a bar with a central circular hole subjected to an axial tension.

A Section Lecture given on 17th February 1955.

From Frocht's "Photoelasticity," published by John Wiley

FIGURE 2. Photoelastic stress pattern of the head of a bar subjected to a tension by means of a single central rivet with a snug fit.



Crown Copyright

FIGURE 3 (*above*). Stress distribution in test-piece end fitting.

Crown Copyright

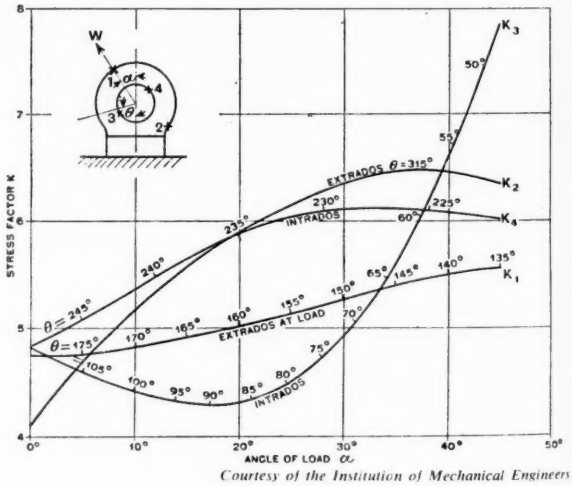
FIGURE 4 (*left*). Stress distribution in B.S.I. sling hook.

Courtesy of the Institution of Mechanical Engineers

FIGURE 5. Hooks after static, impact and fatigue tests.

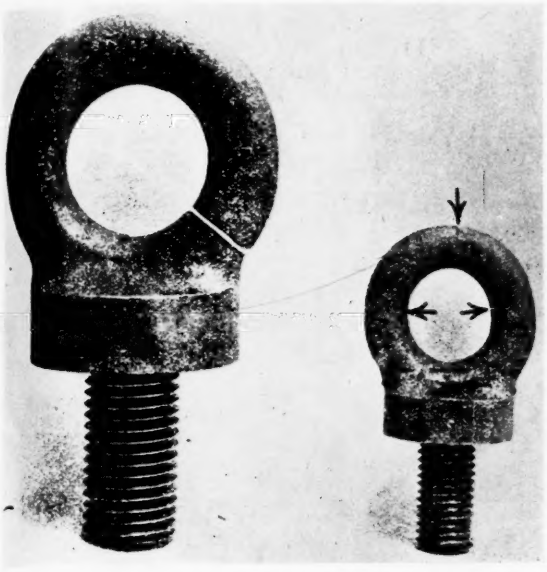
our eye was not entirely "in" and the stress at the point of failure was in fact about 5 per cent. greater than the stress at the horizontal section. In the Liverpool hook the failure was in the right place and in both hooks it occurred after endurances of the order to be expected in relation to the computed stress ranges.)

Another case from the same field is shown in Fig. 6 which shows the stresses developed in eyebolts under direct lift and oblique loading. Fig. 7 shows one eyebolt broken under repeated oblique loading and another broken under repeated direct lift. The computed stresses at the crown at the extrados and at the sides at the intrados in the latter case were about equal and, as can be seen, the eyebolt cracked in all three places.



Courtesy of the Institution of Mechanical Engineers

FIGURE 6. Stresses in eyebolts.

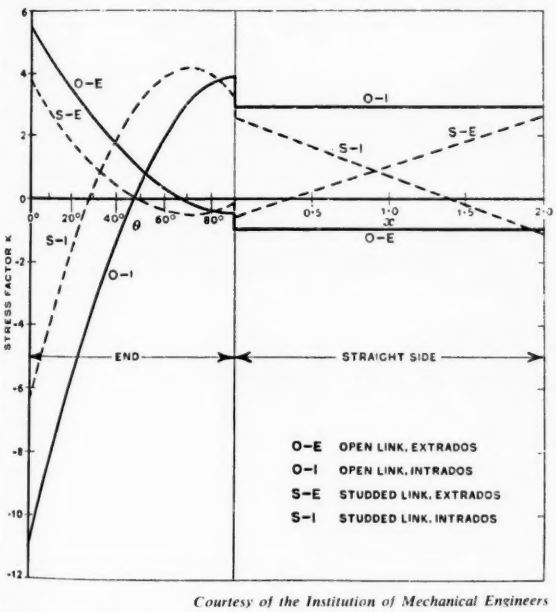


Courtesy of the Institution of Mechanical Engineers

FIGURE 7. Fatigue cracks in eyebolts.

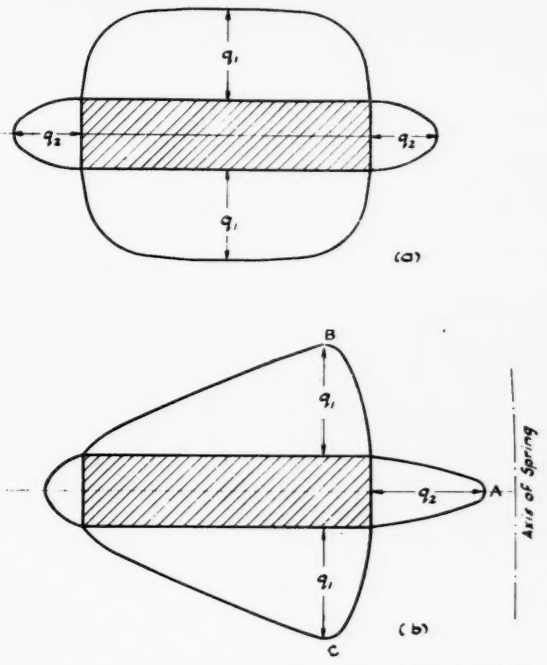
Moreover the endurance was again just about right for the computed stress range.

Figure 8 shows the stress distribution in ordinary and studded chain links and the discontinuity in stress at the end of the parallel portion again illustrates stress concentration superposed on load concentration; the latter is of course due to the bending moment associated with the offset load, while the former—the sudden jump in stress factor—arises from the curvature of the end of the link. We have done



Courtesy of the Institution of Mechanical Engineers

FIGURE 8. Variation of stress round open and studded chain links of same dimensions.



Crown Copyright

FIGURE 9. Distribution of shear stress round a rectangular section under torsion.

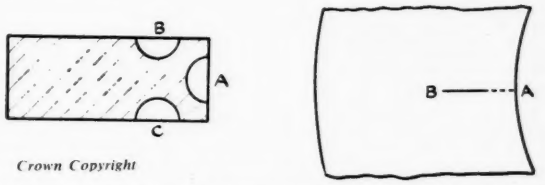
(a) Straight bar.

(*b*) Bar bent into helical spring.

quite a number of fatigue tests on chains, and failure always occurs at about the expected endurance for the computed stress range, either at the crown at the extrados or at the intrados at the end of the straight side. It is to be noticed that we never get fatigue cracks at the intrados at the crown, although the stress range there is far higher than in the other two places; presumably the compressive mean stress is the reason.

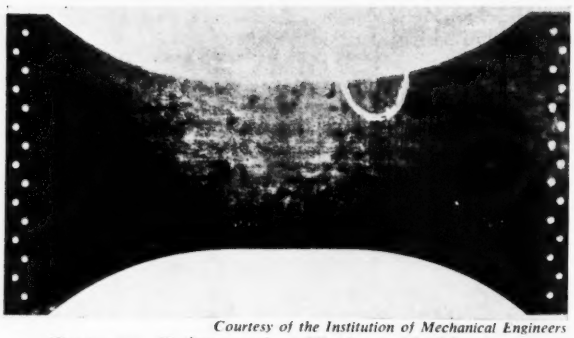
Another example of concentration of stress due to curvature is shown in Fig. 9, which represents the distribution of shear stress over the section of a rectangular section spring. Fig. 10 shows the cracks developed under repeated loading; again the cracks occur in all three regions of highest stress, but it is to be noticed that these cracks propagate on planes of maximum shear stress (whereas those in the lifting gear propagate on planes of maximum tensile stress) and also that they advance only slowly into regions originally subjected to lower stresses.

At this point it may be remarked that much evidence has recently been compiled that cracks often start in



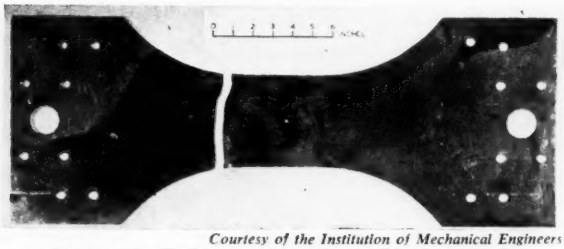
Crown Copyright

FIGURE 10. Positions of cracks in rectangular section springs.



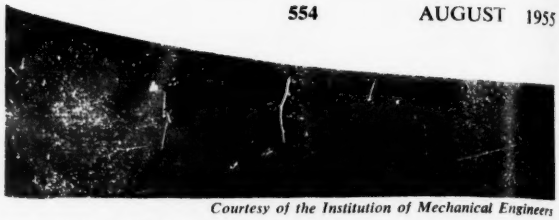
Courtesy of the Institution of Mechanical Engineers

FIGURE 11. Fatigue crack at fillet in a wide thin plate.



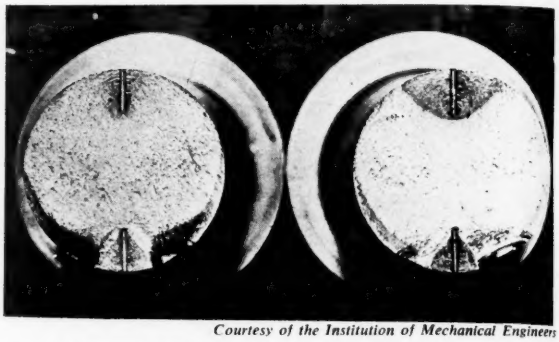
Courtesy of the Institution of Mechanical Engineers

FIGURE 13. Fatigue crack starting at fillet instead of central hole.



Courtesy of the Institution of Mechanical Engineers

FIGURE 12. Detail of a fatigue crack near the fillet in a wide thin plate.



Courtesy of the Institution of Mechanical Engineers

FIGURE 14. Fatigue cracks spreading from small diametral holes.

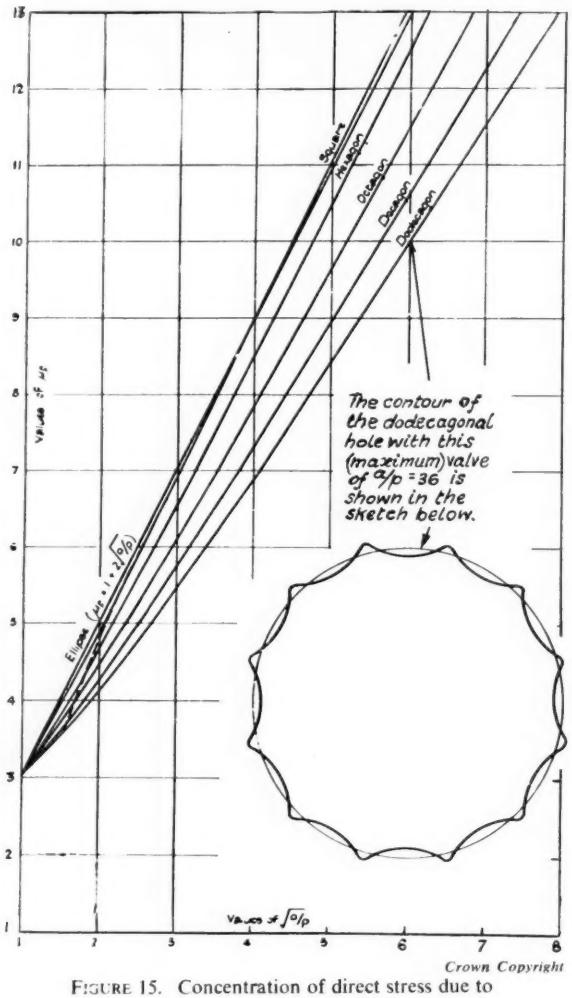


FIGURE 15. Concentration of direct stress due to polygonal holes.

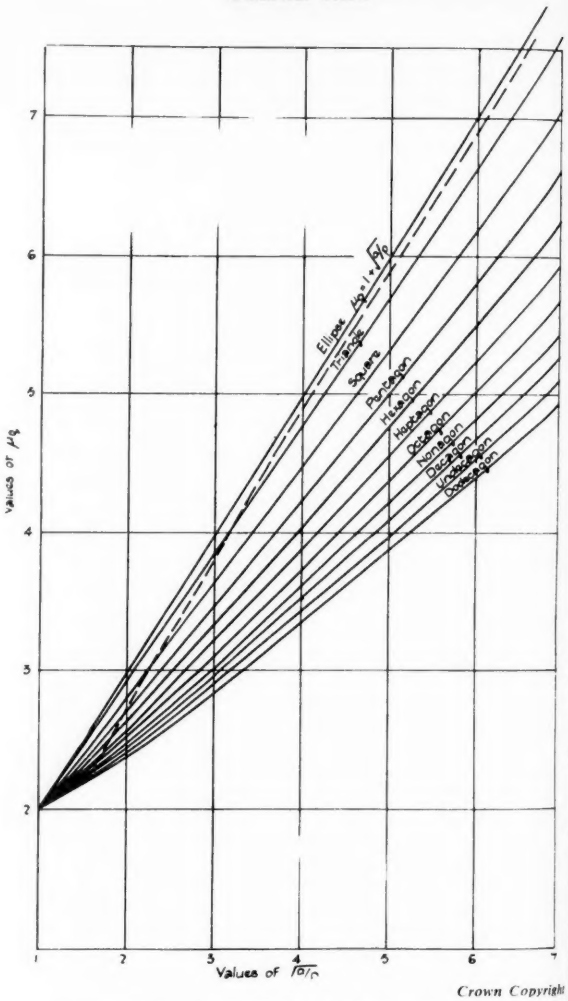
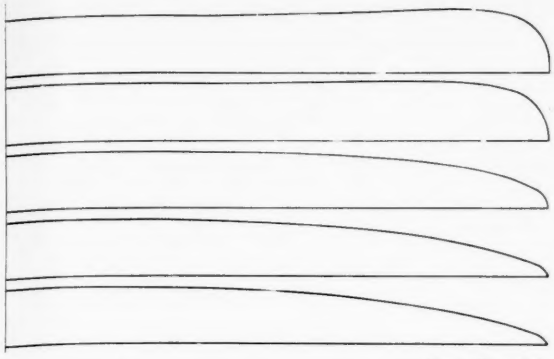


FIGURE 16. Concentration of shear stress due to polygonal holes.



Crown Copyright

FIGURE 17. Forms of deep, narrow groove.

regions of high stress after endurances which are reasonable in relation to the computed range of stress; but that the rate of propagation of such cracks depends very much upon the stress gradient, that is, the rate at which the stress falls off from the point of maximum value; and most remarkably that the cracks often seem blissfully unaware that they themselves ought to be having a tremendous influence on the stress distribution in their neighbourhood.

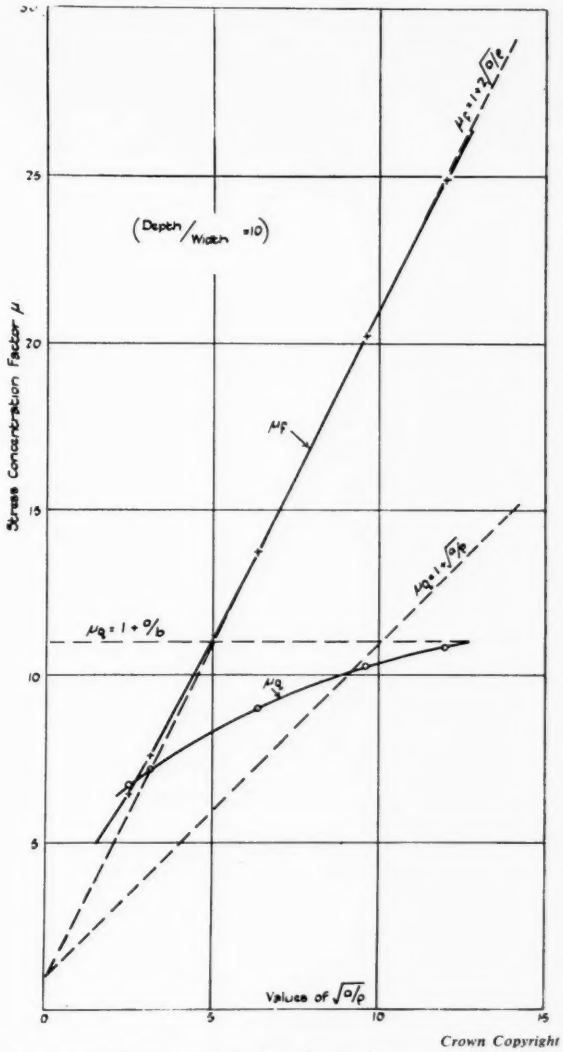
The unerring way in which cracks find regions of high stress is illustrated by Fig. 11 which shows a fatigue crack at a fairly generous fillet in a wide thin plate. Fig. 12 shows the detail of a similar crack: the stress concentration was only about 1·20 but the gradient was not very steep, so that this particular crack formed away from the edge. Fig. 13 illustrates how this gradient effect can favour cracking at a fillet with a stress factor of 1·2, instead of at a hole where the stress factor was 3 but the gradient was very much steeper. The stress gradient effect is illustrated again in Fig. 14 which shows cracks which have spread only rather slowly from small diametral holes.

From this brief survey of the effect of stress concentration in causing fatigue fracture it will appear that we may need to know not only the maximum stress at the discontinuity of section, but also something about the rate of change of the stress in this neighbourhood. That question is bound up with the general problem of stress analysis for discontinuities more complex than the simple holes and fillets which I have dealt with so far.

Special Two-Dimensional Boundaries

General analysis in two dimensions is not particularly difficult and some results are shown of an investigation relating to holes of quite a variety of forms in an infinite plate. For holes in plates of finite width the photoelastic method would have to be used, but the general theory suffices to indicate the effect of form of hole. Fig. 15, for instance, shows the concentration factors under uniaxial tension of a wide range of polygonal holes, of which one is illustrated. On this figure the line $1+2\sqrt{(a/\rho)}$ is marked; this represents the stress factor for an elliptical hole, and it will be seen that it affords a surprisingly good approximation to the true factor, even for holes of quite different form. Fig. 16 shows the factors for the same holes under shear parallel to the axis of the hole; here it will be seen that the formula $\mu_q=1+\sqrt{(a/\rho)}$ for an elliptical hole is far wider of the mark. There is one special class of polygonal holes which have $\rho=\frac{1}{2}a$ alternating with $\rho=\infty$, such as a square with flat sides but rounded corners. If the number of sides of such a hole is increased without limit, the hole becomes indistinguishable from a true circle, yet the formula $1+2\sqrt{(a/\rho)}$ would indicate a stress factor $1+2\sqrt{2}$, whereas the formula on which the figure is based indicates the obviously correct value 3. The important conclusion here is that abrupt changes of curvature in themselves do not affect stress concentration.

Figure 17 shows a range of rather deep grooves having various curvatures at their ends, and Fig. 18 shows that again $1+2\sqrt{(a/\rho)}$ affords a good estimate



Crown Copyright

FIGURE 18. Stress concentrations due to deep, narrow grooves.

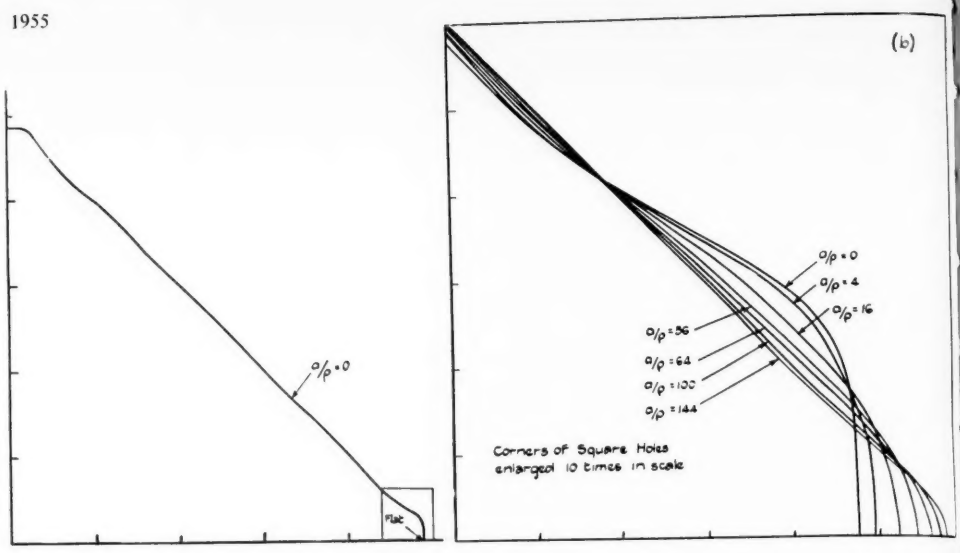
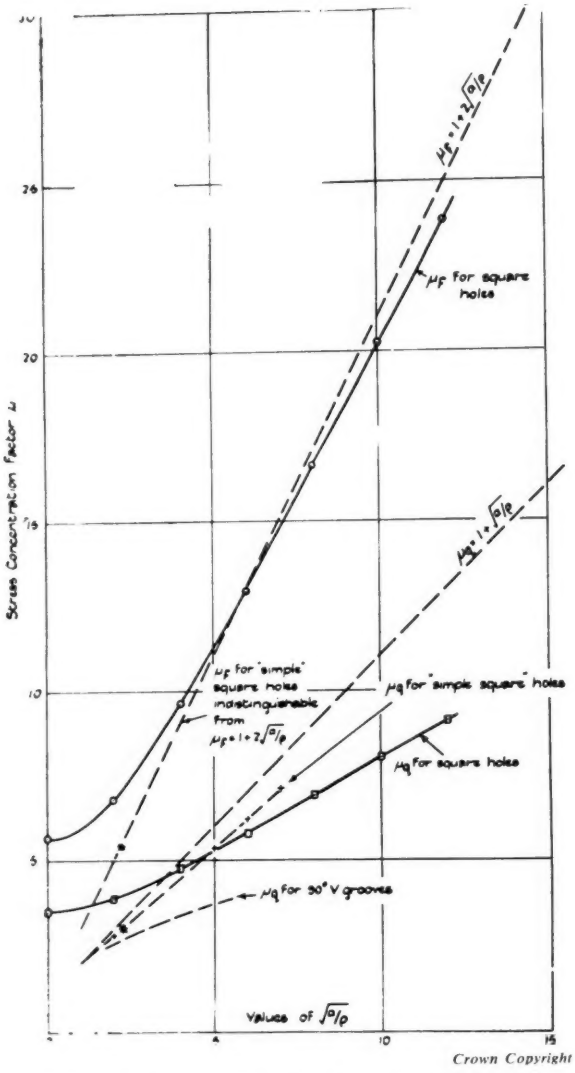


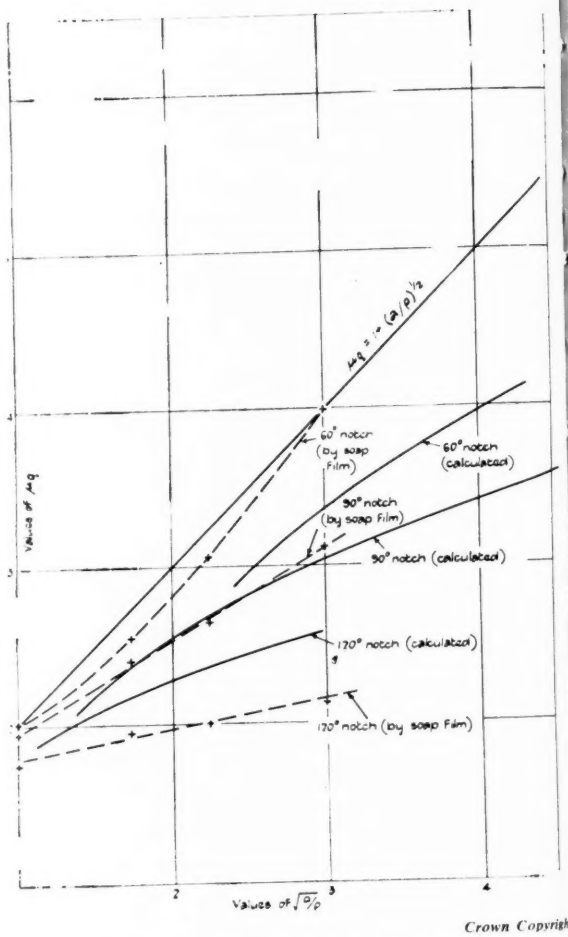
FIGURE 19. Forms of square holes.

Crown Copyright


Stress Concentration Factor μ
Values of √a/ρ
μf = 1 + 2√a/ρ
μf for square holes
μq = 1 + √a/ρ
μf for "simple" square holes indistinguishable from μf = 1 + 2√a/ρ
μq for "simple square" holes
μq for square holes
μq for 90° V grooves

Crown Copyright

FIGURE 20. Stress concentrations due to square holes and 90° Vee grooves.



Crown Copyright

FIGURE 21. Concentration of shear stress at Vee grooves.

of the stress concentration under direct stress, whereas $1+\sqrt{(a/\rho)}$ for the shear cases is far less accurate. The same point is further illustrated by Fig. 19 which shows a range of square holes and Fig. 20 which shows the corresponding stress factors. The curve of μ_q for 90° Vee grooves shown on this figure was computed by a different method; that this curve should lie appreciably below that for square holes is surprising, because the form of the groove differs very little from that of the square hole. This difference is difficult to understand, but the V-groove curve agrees very well with values obtained by Griffith using the soap film method (Fig. 21).

The formal solution for these special boundaries is easy enough but the detailed working is tricky. On the other hand the solution of the torsion problem for any boundary is entirely straightforward. For instance, Fig. 22 shows the stress distribution in a hollow lobed shaft, and Fig. 23 the stress in a splined shaft. Such examples could be multiplied without limit, but Fig. 24 shows an application which is at once far less elegant and far more useful. This was an early check on the general method by application to a square shaft. Using a rather simple means of representation I finished up with a shaft with wobbly sides and well rounded corners. However, I was a bit disappointed when I found the stress factor varying from about 1·1 to 1·6 along the middle of the side, instead of keeping close to the value 1·35 which Timoshenko assured me was the right answer. Then I had the happy idea to "correct" the computed values at each trough by dividing by $1+\sqrt{(h/r)}$, where h is the depth of the local trough and r is its radius of curvature, and the still happier idea to divide the value at each crest by $1-\sqrt{(h/r)}$, where h is the height of the crest and r its radius of (convex) curvature. By this means the factor

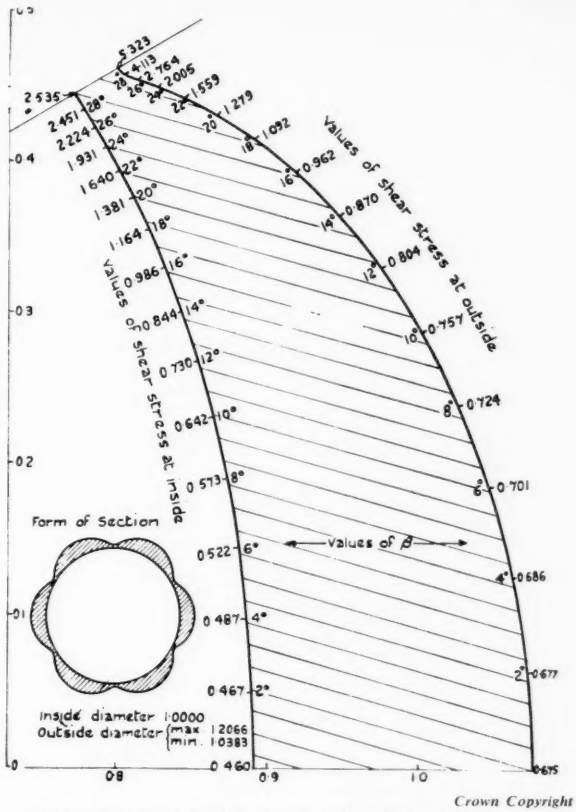


FIGURE 22. Distribution of stress in a hollow lobed shaft.

was brought into line with Timoshenko. Fig. 25 shows the same procedure applied to a round shaft with a single flat. In this case there is no exact solution for

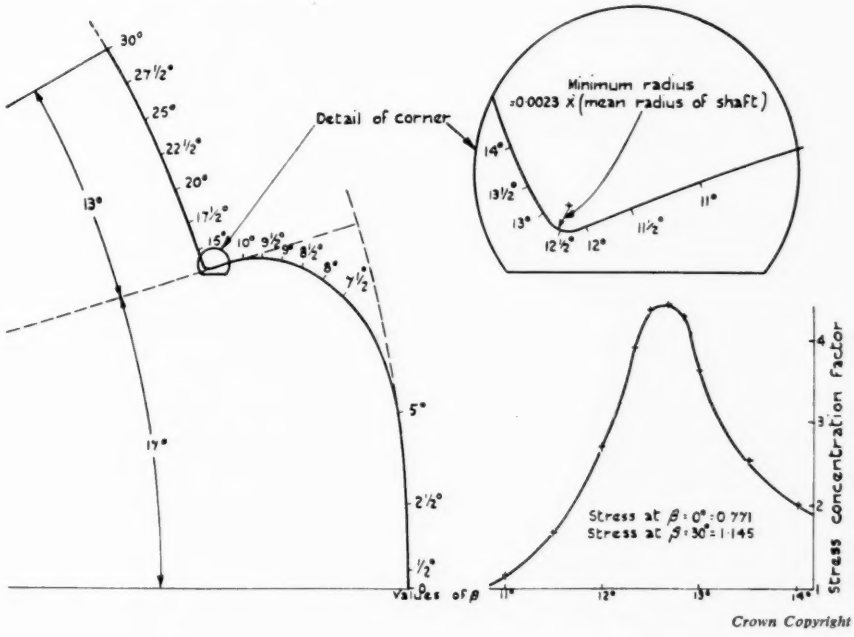


FIGURE 23. Stress distribution round a sharp corner in a six-splined shaft.

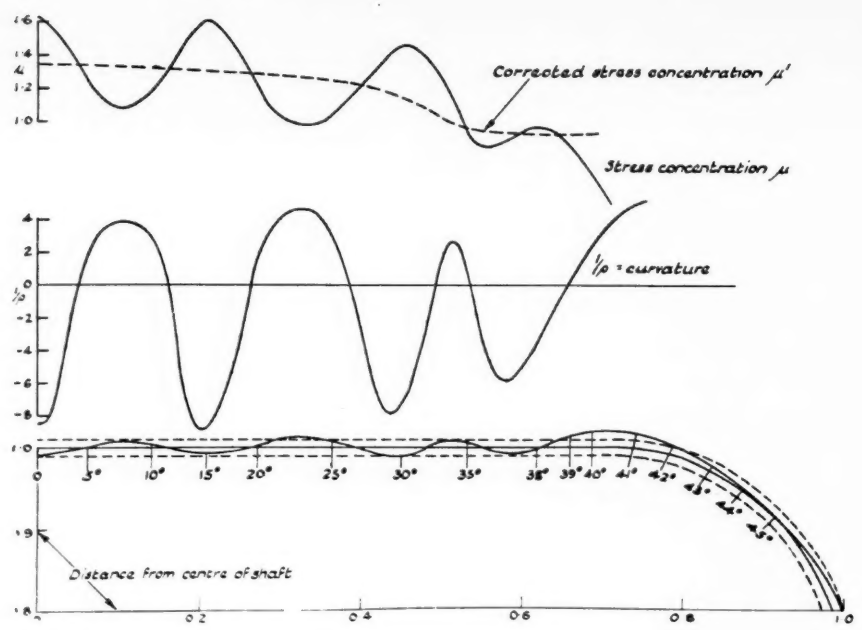


FIGURE 24. Distribution of shear stress along the side of a square shaft under torsion.

Crown Copyright

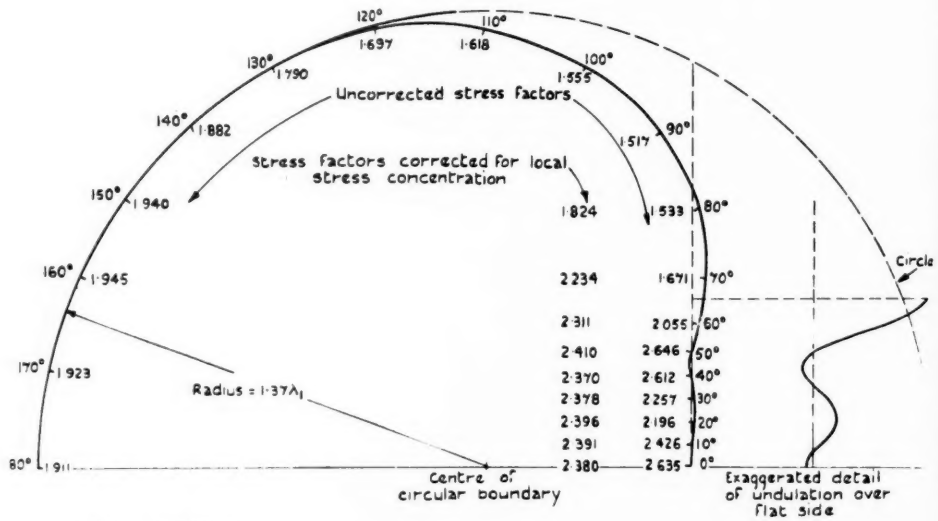

Uncorrected stress factors
Stress factors corrected for local stress concentration
Radius = 1·37A1
Centre of circular boundary
Circle
Exaggerated detail of undulation over flat side

FIGURE 25. Distribution of shear stress in a round bar with a flat.

Crown Copyright

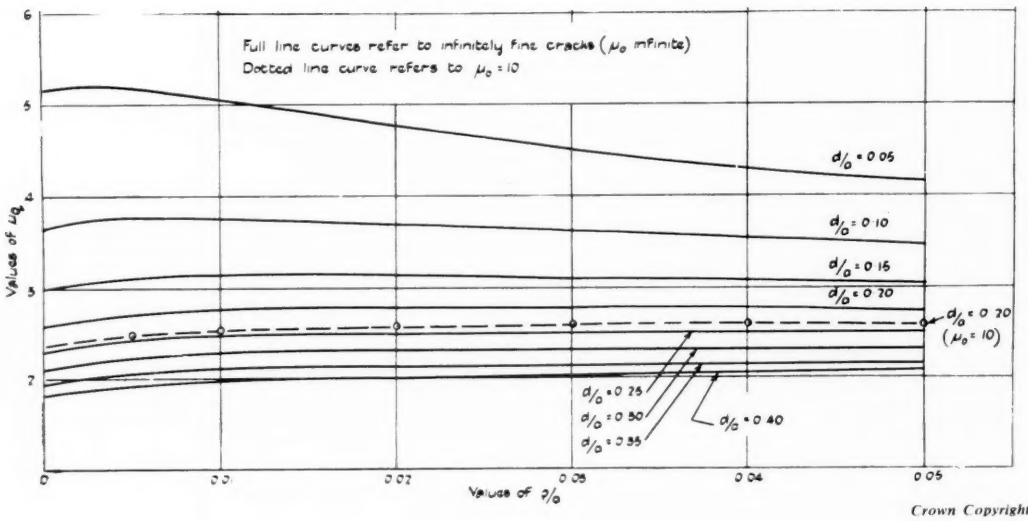
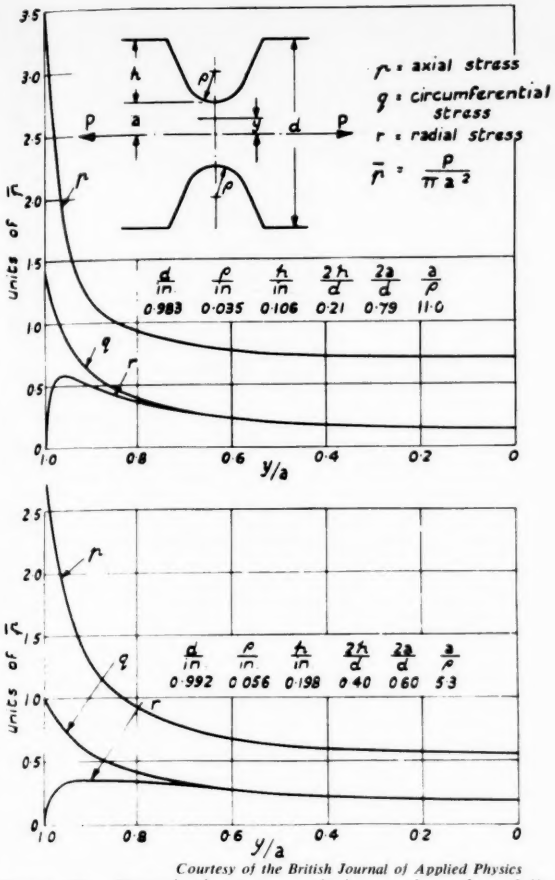


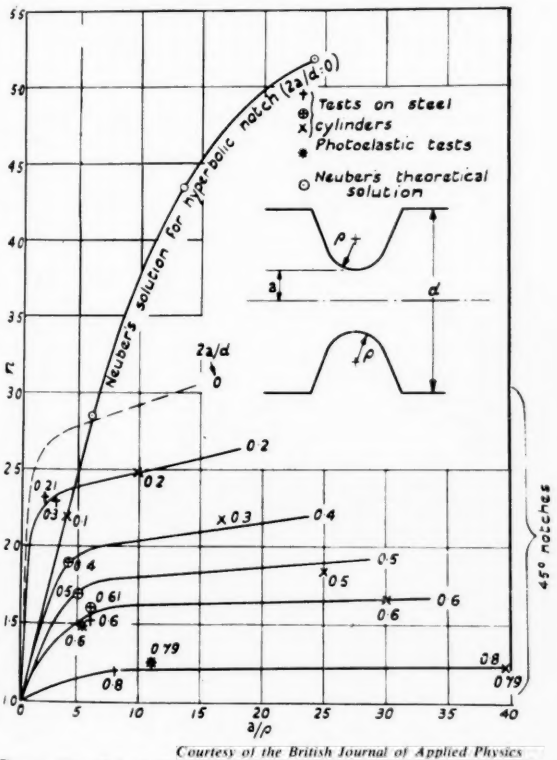
FIGURE 26. Effective stress concentration factors under shear for material containing flaws.

Crown Copyright


f = axial stress
g = circumferential stress
r = radial stress
f̄ = P/πa²
units of f
y/a
d/in ρ/in h/in 2h/d 2a/d a/ρ
0·983 0·035 0·106 0·21 0·79 11·0
0·992 0·056 0·198 0·40 0·60 5·3

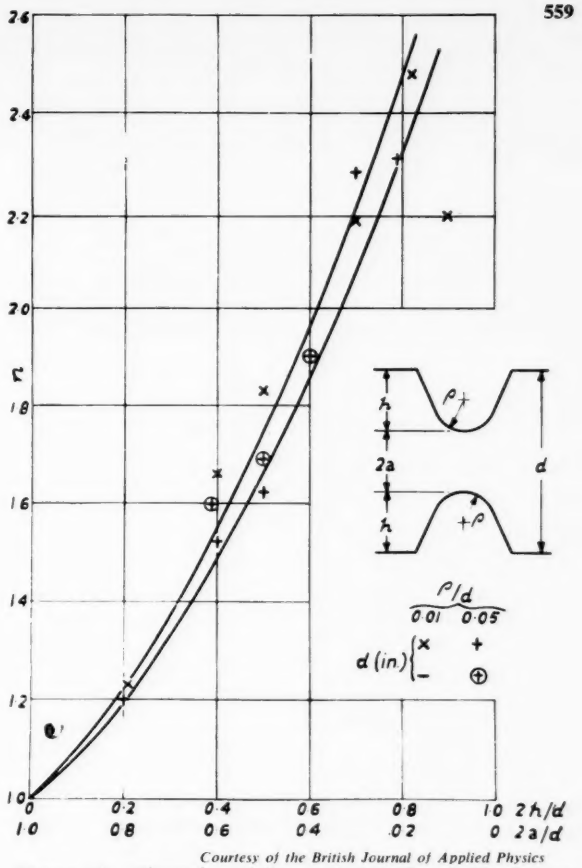
Courtesy of the British Journal of Applied Physics

FIGURE 27. Photoelastic stress analysis on circumferentially notched cylinders in tension. Principal stresses at notch plane.



Courtesy of the British Journal of Applied Physics

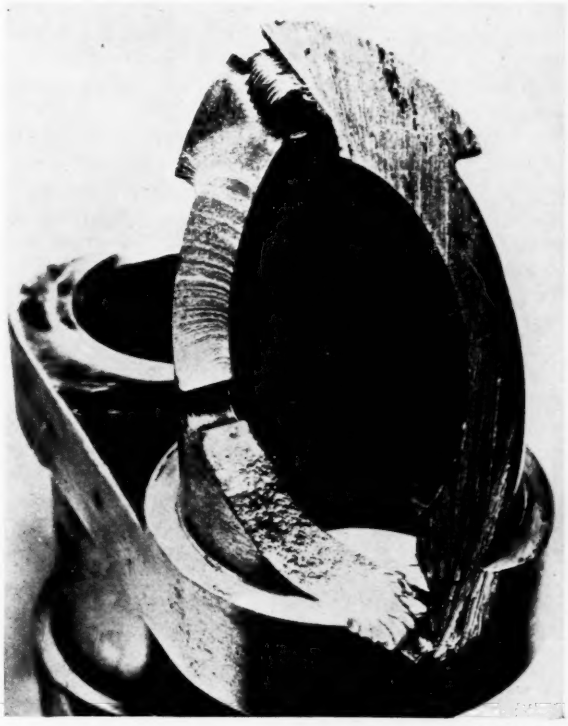
FIGURE 29. Triaxiality ratio (n) for circumferential Vee notches in tension. (Figures against points are values of $2a/d$.)



Courtesy of the British Journal of Applied Physics

FIGURE 28. Circumferentially notched steel cylinders in tension. Triaxiality factor, n, expressed as

$$\frac{\text{1 per cent. proof stress on notched specimen}}{\text{1 per cent. proof stress on plain specimen}}$$



Crown Copyright

FIGURE 30. Fatigue failure of crankshaft starting at oil hole.

Courtesy of the Institution of Mechanical Engineers

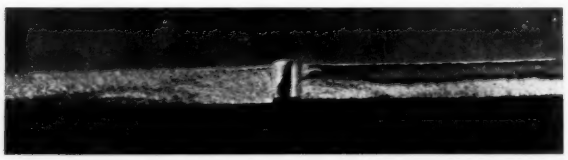
FIGURE 31. Crack propagating from a small central hole in a wide, thin plate.

comparison but I think that it is clear that the corrected values must be pretty near the mark.

Practical Aspects

One of the main reasons for this general investigation of form of hole was to determine the stress concentration due to a small crack or flaw at the root of a notch. If the two separate factors for the flaw and for the notch were always to be multiplied together the presence of flaws would not affect notch sensitivity; but if the two were not always multiplicative, the effect would be to reduce notch sensitivity because notch and flaw together might cause only slightly higher concentration than the flaw alone. This application of the general analysis proved rather difficult but the results shown on Fig. 26 for the factor in shear are probably not far from the truth. I have plotted the apparent stress factor, that due to notch and flaw together divided by that due to flaw alone. To explain limitation of notch sensitivity in this way it is necessary to postulate depths of flaw d appreciable in relation to notch depth a. On the other hand, the effect is to render the stress factor very insensitive to notch root radius, which was the experimental finding I was trying to explain. The main objection is of course that such long cracks ought to be discernible by some means or other, but there are other serious objections to the flaw hypothesis which it will not be easy to surmount. However it seems quite possible, in relation to recent observations on cracks which do not propagate, that the old Griffith idea may shortly gain a new lease of life in a modified form.

The theoretical results referred to all relate to holes in infinite plates or to cases of shear or torsion of long bars of uniform section. For other cases photoelasticity has to be used and for truly three-dimensional problems the frozen stress technique is needed. My colleague



Courtesy of the Institution of Mechanical Engineers

FIGURE 32. Fatigue crack changing from transverse plane to two inclined planes.

Courtesy of the Institution of Mechanical Engineers

FIGURE 33. Fatigue crack changing from two inclined planes to plane of maximum shear.

Brown has recently applied this method to two cases of round bars with circumferential V notches under tension across the notch. The results are shown in Fig. 27. It will be seen that the sharp, shallow notch causes the higher stress concentration but that the stress falls off far more rapidly, so that the stress distribution in the core of the bar under the notch is not very different in the two cases. The figure shows also the radial and hoop stresses under the notch, and the shear stress in the core is represented by the difference between the p and r curves. It will be seen that the ratio of p to the difference $(p-r)$ is greater for the deeper notch. This ratio—if we can judge its average value—is important because if p reaches the limiting value for cleavage before $(p-r)$ reaches the limiting value for plastic deformation, brittle fracture will result.

Since photoelastic investigation by the frozen stress method is a long laborious task, Brown took a short cut to the desired answer by testing a number of notched pieces of mild steel. In Fig. 28 he has defined the triaxiality ratio average p/average $(p-r)$ as the ratio of the one per cent. proof stress for the notched bar to that for the plain bar, which is as good a definition of triaxiality ratio as any other. The effect in these tests of the root radius of the notch was quite unimportant; it was the depth of the notch that mattered. It will be seen that a notch leaving a core only 20 per cent. of the diameter of the bar gave a triaxiality ratio of 2·5. This final composite diagram (Fig. 29) shows the conclusions drawn both from the tests on mild steel and from the photoelastic analyses with a satisfactory measure of agreement between the two approaches. Neuber's results appear rather different, but comparison between the infinitely deep notch and notches of finite depth is very difficult.

In conclusion I want to return once more to the

Courtesy of the Institution of Mechanical Engineers

FIGURE 34. Similar change of mode of crack propagation at opposite side of test piece (*see* Fig. 31).

most interesting problem of crack propagation. Fig. 30 shows a fatigue crack developed from an oil hole in a crankshaft. The crack propagated along the plane of maximum tension as a result of alternating torsional stresses arising probably from torsional vibration. This might be described as a classic example, in the sense that everyone who feels that his upbringing ought to have been classical, will pretend that he completely understands it. In contrast, Fig. 31 is a crack developing from a small hole in a wide thin plate of aluminium alloy under repeated tension. The cracks propagated almost equally on the two sides—a most encouraging advertisement of the testing technique—suddenly both advancing cracks changed their mode of propagation, speeded up their rate of progress and cut the piece in two. This final stage could be described as failure under the static mean load, but still I believe that even in this stage the range of load was playing a part—a compromise description might be "static failure under the maximum tensile load in the cycle."

However, the change of mode of propagation apparent in this figure is not the first. There was another change earlier which appears in Fig. 32. The crack from the side of the hole starts to propagate on the plane of maximum tension, proceeds only a few millimetres in that manner and then transfers rather abruptly to a mode, which pays some, but not entire, regard to shear. The cattle shed roof is quite characteristic and the angle of pitch is about $\pm 30°$ from the horizontal or transverse plane; precise measurement of the angles is difficult but they are certainly far removed from 45°. The second change of mode of propagation apparent on the previous figure is shown in Fig. 33 and again in Fig. 34. The change here is very abrupt indeed from cattle shed roof to a single plane of shear at about 45°.

There is certainly no need at the present time to urge that more attention be paid to crack propagation—we are all watching cracks extend or not extend for all we are worth; but if I may draw my own moral from my own talk I should urge that more attention should be paid to stress distribution round the advancing crack. At present the common practice is just to remark that the stress at the tip is very high or infinite and to leave it at that: we should do better.

Acknowledgments

Acknowledgment is made by the author to the Controller, Her Majesty's Stationery Office for permission to reproduce Figs. 15-26, which are Crown Copyright and are reproduced from R. & M. 2704; and to the Institution of Mechanical Engineers for Figs. 5-8, 11-14; the Institute of Physics for Figs. 27-29, which are from a paper by A. F. C. Brown published in the "British Journal of Applied Physics" 1954; and the Institution of Mechanical Engineers for Figs. 31-34. Figs. 1 and 2 are reproduced from "Photoelasticity" by Max Frocht, published by John Wiley & Sons Inc., to whom acknowledgment is also made.

TECHNICAL NOTES

Contributions to this Section of the JOURNAL *will be eligible for Journal Premium Awards and will normally be published within two months of being received.*

The Effect of Curvature on the Centre of Shear

by
W. JOHNSON
(Department of Mechanical Engineering, University of Sheffield)

A CONSIDERABLE AMOUNT of attention has been devoted to the study of the position of the centre of shear for straight beams but there appears to have been little or no consideration of this position for beams of which the centre line is either slightly or heavily curved. The reasons are probably twofold. Firstly, no demand for such an investigation has been made by designers, and secondly, and probably not least, any very general inquiry into this problem will lead to complex expressions which would not only be diffiult and tedious to evaluate but will also be of uncertain accuracy.

This problem of taking beam curvature into account starts from the point where it normally ends for most mechanical engineers, who usually require to estimate the bending stresses in members of large curvature such as hooks and chain-links. Now the classical theory of elasticity applied to these latter members has yet to be seen to conform with experimental results to such a high degree as makes the added labour attending such methods worth while. It has been found that "engineers' theory" applied to highly curved bars, *i.e.* such as Winkler's Theory, when applied to these particular cases, provides results that are quite satisfactory, and it is on the knowledge of this that the approach in Section 2 is justified. No claim is made for an exact prediction of the results of possible experimental investigation, but it is believed that both the trend of the theeoretical results and their numerical values will provide a useful indication of what may be expected.

Two simple thin-walled sections are considered, the channel and the semi-circle. The thickness of the material is assumed to be negligible in comparison with other dimensions and the influence of possible alterations in the shape of the sections under the action of the applied loads is also neglected, as is the influence of lateral pressure between fibres in the case of the semi-circular section. From these and other less important assumptions, expressions are derived for the position of the centre of shear when a moment acts in a plane parallel to the centre line of the member and when normal and shear forces are also acting across the section. The particular cases of a plane vertical quadrant *encastré* at one end and carrying a simple vertical concentrated load at the other is then investigated for each type of section and a comparison of the results is made.

Received 6th May 1955.

NOTATION

M	moment on section about the point where the centre-line cuts it
H	normal force on section
V	shear force across section
p	normal stress on section due to bending
p_t, p_c	tensile or compressive bending stress
q	shear stress
t	thickness of material
A	cross-sectional area
Ak^2	"modified area" of section
e	shear centre distance

1. CHANNEL SECTION

1.1. *General Case*

Consider a channel cross section having its flanges unequal in width and thickness, as shown in Fig. 1, and a centre-line which is the arc of a circle of radius R, *see* Fig. 2. Across a section such as AB the moment and forces are as indicated in Fig. 3.

For force equilibrium normal to the section, neglecting the bending stresses in the web

$$p_t\, at_1 - p_c bt_2 = H \quad . \quad . \quad . \quad (1)$$

and for moment equilibrium

$$h(p_t\, at_1 + p_c bt_2) = M \quad . \quad . \quad (2)$$

Substituting from (1) in (2) gives

$$2p_c bt_2 = \frac{M}{h} - H$$

and thus

$$2hbt_2 \frac{\partial p_c}{\partial \theta} = \frac{\partial}{\partial \theta}(M - hH) \quad . \quad . \quad (3)$$

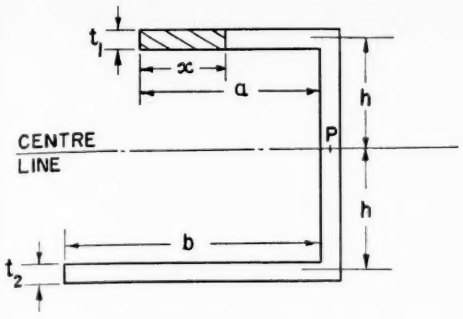


FIGURE 1. Channel section.

Consider next an element from the flange of width x, *see* Fig. 1, whose length is $R_0\delta\theta$, compare Fig. 2. For force equilibrium, *see* Fig. 4, a shear force on the back face of the block is required, given by

$$\frac{\partial p}{\partial \theta}\,\delta\theta x t = q t \delta\theta R_0$$

or

$$q = \frac{x}{R_0}\frac{\partial p}{\partial \theta}$$

In particular

$$q_c = \frac{x}{R_1}\frac{\partial p_c}{\partial \theta} \qquad . \quad . \quad (4a)$$

and

$$q_t = \frac{x}{R_0}\frac{\partial p_t}{\partial \theta} \qquad . \quad . \quad (4b)$$

The complementary shear stresses to these quantities give rise to the 'shear flow' round the section. Utilising (3),

$$q_c = \frac{x}{2R_1 h b t_2}\frac{\partial}{\partial \theta}(M - hH) = xk_c \text{ say} \qquad (5a)$$

and

$$q_t = \frac{x}{2R_0 h a t_1}\frac{\partial}{\partial \theta}(M + hH) = xk_t \text{ say} \qquad (5b)$$

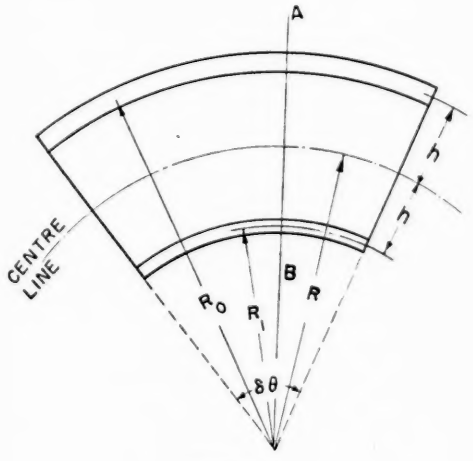


FIGURE 2.

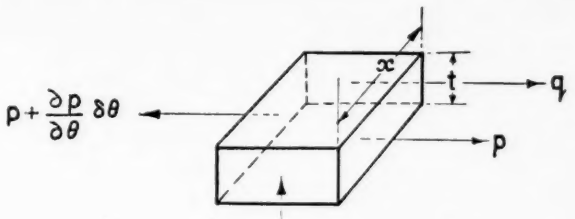


FIGURE 4.

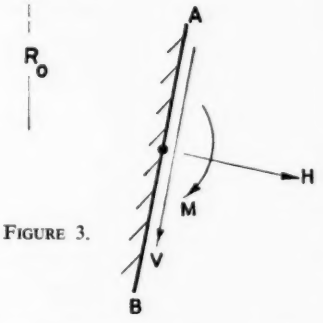


FIGURE 3.

The moments of the complementary shear stresses about P, which are equidistant from the flanges, give

$$M_t = k_t \frac{a^2}{2} h t_1 \text{ and}$$

$$M_c = k_c \frac{b^2}{2} h t_2.$$

The transverse shear force V is required to act at a distance e from P such that

$$Ve = M_t + M_c = \frac{h}{2}\{k_t a^2 t_1 + k_c b^2 t_2\}.$$

On substituting and rearranging,

$$e = \frac{1}{4V}\left[\left(\frac{a}{R_0} + \frac{b}{R_1}\right)\frac{\partial M}{\partial \theta} + h\left(\frac{a}{R_0} - \frac{b}{R_1}\right)\frac{\partial H}{\partial \theta}\right] \qquad (6)$$

1.2. *A Particular Case*

In particular the case shown in Fig. 5 is investigated when $a = b$, and for the loading shown. Here,

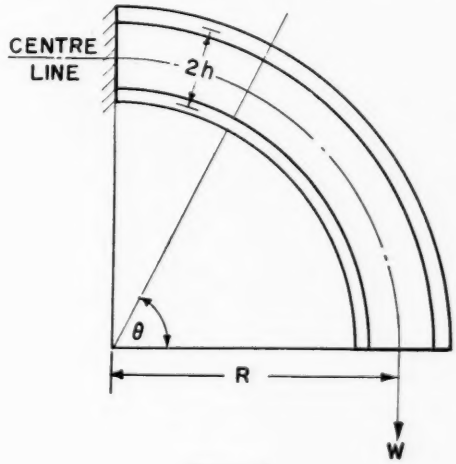


FIGURE 5.

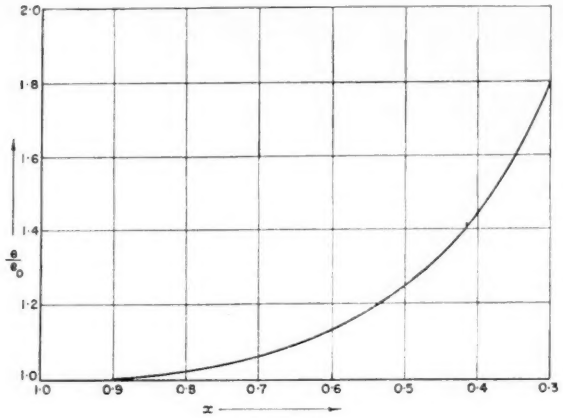


FIGURE 6.

$$M = WR(1-\cos\theta) \quad \text{and} \quad \frac{\partial M}{\partial \theta} = WR\sin\theta$$

$$H = W\cos\theta \quad \text{and} \quad \frac{\partial H}{\partial \theta} = -W\sin\theta$$

$$V = W\sin\theta.$$

Substituting in (6) we eventually obtain on reduction, that

$$e = \frac{a}{2}\frac{1+x^2}{2x}$$

where

$$x = \frac{R_1}{R_0}.$$

When $x=1$, a straight beam is being considered and then $e=a/2=e_0$, say. Fig. 6 shows the variation of e/e_0 with x. Alternatively in terms of R and h, if $h/R=v$,

$$\frac{e}{e_0} = \frac{1-v^2}{1+v^2} \quad . \quad . \quad . \quad (7)$$

The variation of e/e_0 with v is shown in Fig. 7.

2. SEMI-CIRCULAR SECTION

2.1. Figure 8 shows the section of radius r. Again the curvature of the member is assumed to be constant along its length. Now it can be shown from ordinary engineering theory[1] that

$$p = \frac{M}{AR}\left(1+\frac{R^2}{k^2}\frac{y}{R+y}\right)+\frac{H}{A} \quad . \quad (8)$$

The total force on that portion of the section covered by angle ϕ is F and

$$F = \int_0^{\phi} pt + d\phi$$

or $$\frac{F}{rt} = \int_0^{\phi}\left[\frac{1}{A}\left(\frac{M}{R}+H\right)+\frac{MR}{Ak^2}\frac{r\cos\phi}{R+r\cos\phi}\right]d\phi$$

i.e. $$\frac{F}{rt} = \phi\left(\frac{M}{AR}+\frac{H}{A}+\frac{MR}{Ak^2}\right) -$$

$$-\frac{MR^2}{Ak^2}\frac{2}{\sqrt{(R^2-r^2)}}\tan^{-1}\left(\sqrt{\left(\frac{R-r}{R+r}\right)}\tan\phi/2\right)$$

when $R>r$.

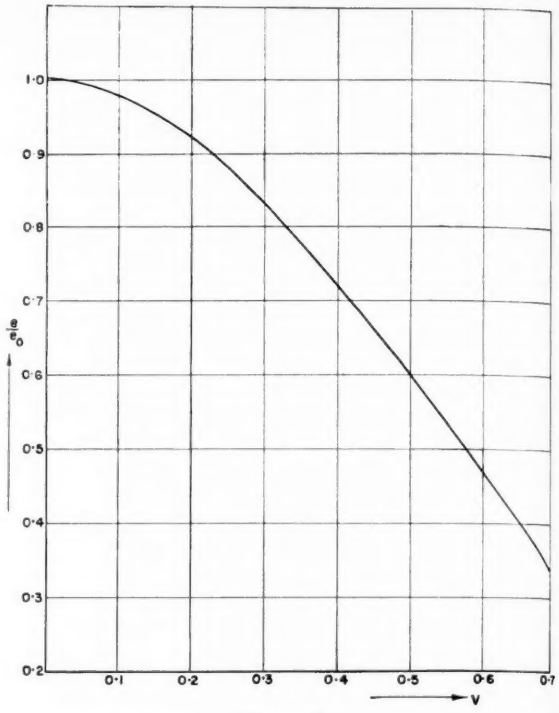


FIGURE 7.

Thus

$$\frac{1}{rt}\frac{dF}{d\theta} = \phi\left[\frac{1}{AR}\frac{\partial M}{\partial\theta}+\frac{1}{A}\frac{\partial H}{\partial\theta}+\frac{R}{Ak^2}\frac{\partial M}{\partial\theta}\right] -$$

$$-\frac{2R^2\dfrac{\partial M}{\partial\theta}}{Ak^2\sqrt{(R^2a^2)}}\tan^{-1}\left(\sqrt{\left(\frac{R-a}{R+a}\right)}\tan\phi/2\right) \quad (9)$$

Now, also by considerations similar to those applied in connection with Fig. 4, and except that forces and not stresses have to be considered immediately, the shear stress across section OQ is given by

$$dF = qt(R+y)\,d\theta$$

or $$\frac{1}{rt}\frac{dF}{d\theta} = \frac{q}{r}(R+y) \quad . \quad (10)$$

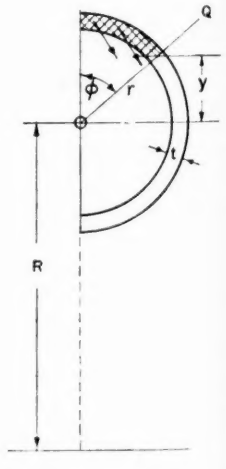


FIGURE 8.
Semi-circular section.

Equating (9) and (10)

$$q=\frac{r}{A(R+r\cos\phi)}\left[\left(\frac{\phi}{R}+\frac{\phi R}{k^2}-\right.\right.$$

$$\left.\left.-\frac{2R^2}{k^2\sqrt{(R^2-r^2)}}\tan^{-1}X\right)\frac{\partial M}{\partial\theta}+\phi\frac{\partial H}{\partial\theta}\right] \quad (11)$$

where $$X=\sqrt{\left[\frac{(R-a)}{(R+a)}\right]}\tan\phi/2$$

The centre of shear is now given by

$$e=\frac{r^2t}{V}\int_0^\pi q\,d\theta, \quad . \quad . \quad . \quad (12)$$

where e is measured from the centre of the semi-circle.

Substituting from (11) in (12) and simplifying,

$$e\frac{AV}{r^3t}=\left[\left(\frac{1}{R^2}+\frac{1}{k^2}\right)\frac{\partial M}{\partial\theta}+\frac{1}{R}\frac{\partial H}{\partial\theta}\right]I_1-$$

$$-\frac{2R}{h^2\sqrt{(R^2-r^2)}}\frac{\partial M}{\partial\theta}I_2$$

where $$I_1=\int_0^\pi\frac{\phi}{1+\frac{r}{R}\cos\phi}\,d\phi$$

and $$I_2=\int_0^\pi\frac{\tan^{-1}X}{1+\frac{r}{R}\cos\phi}\,d\phi=\frac{\pi^2}{4\sqrt{\left(1-\left(\frac{r}{R}\right)^2\right)}}.$$

Writing $r^3t\,.\,k(v)$ for Ak^2, where $v=r/R$, substituting for I_2 and simplifying further,

$$\frac{e}{r}=\frac{1}{Vk(v)}\left[\frac{1}{r}\frac{\partial M}{\partial\theta}\left\{\left(\frac{v^2}{\pi}+\frac{1}{k(v)}\right)I_1-\frac{\pi^2}{2k(v)(1-v^2)}\right\}+\right.$$

$$\left.+\frac{I_1v}{\pi}\frac{\partial H}{\partial\theta}\right]. \quad . \quad (13)$$

It may be shown that, since

$$Ak^2=\int\frac{Ry^2}{R+y}\,dA$$

then $$\frac{Ak^2}{r^3t}=\frac{\pi}{v^2}\left[\frac{1}{\sqrt{(1-v^2)}}-1\right]=k(v)$$

and, if v is small,

$$\frac{Ak^2}{r^3t}=\frac{\pi}{2}\left(1+\frac{3}{4}v^2\right).$$

If r is finite and $R\longrightarrow\infty$ then $v\longrightarrow 0$ and $Ak^2\longrightarrow\pi/2r^3t$, *i.e.* Ak^2 becomes the second moment of area of the section about the centre-line.

2.2. Applying the result of (13) to the particular situation treated above for the channel section, shown in Fig. 5, on reduction,

$$\frac{e}{r}=\frac{1}{vk(v)}\left[I_1-\frac{\pi^2}{2(1-v^2)}\right].$$

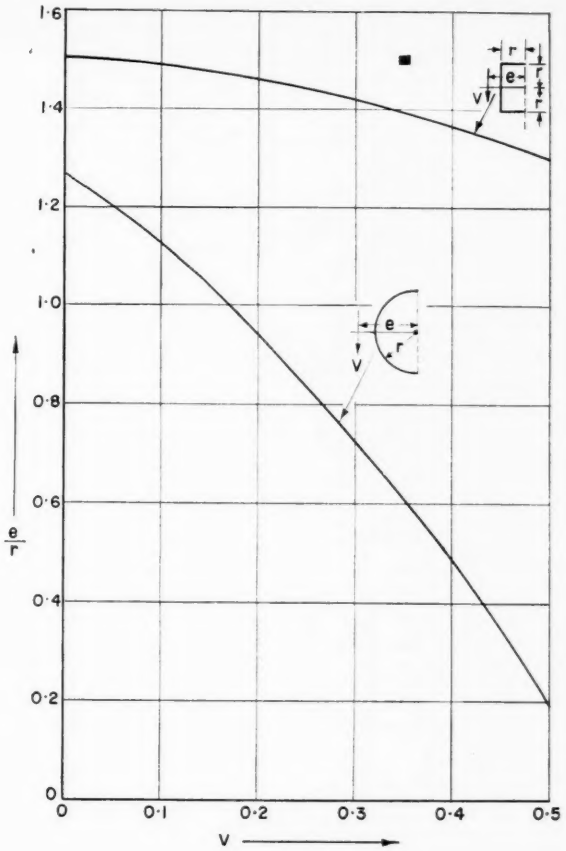


FIGURE 9.

The variation of e/r with v is shown in Fig. 9. It may be noted that when $v\longrightarrow 0$, $\frac{e}{r}\longrightarrow\frac{4}{\pi}=\frac{e_0}{r}$, say.

3. CONCLUSION

In Fig. 9 a comparison is also made of the variation of e/r with v for both sections, when h and a for the channel sections are each equal to r. Two points emerge:

(*i*) that in both cases the tendency is for the centre of shear to move from the outside to the inside of the section as the radius of curvature decreases;

and (*ii*) the rate of this movement in the case of the semi-circular section is much larger than in the case of the channel section.

ACKNOWLEDGMENTS

The author wishes to acknowledge valuable discussions with Mr. E. P. Bernat, F.R.Ae.S.

REFERENCE

CASE, J. (1925). *Strength of Materials*, Edward Arnold & Co., 1925.

Buckling of Rectangular Plates under Bi-Axial Compression

by

J. S. PRZEMIENIECKI, B.Sc., Grad.R.Ae.S.

(Structures Department, The Bristol Aeroplane Co. Ltd.)

THE problem of the buckling under bi-axial compression is considered for flat rectangular isotropic plates with simple edge conditions and no lateral restraint. The buckling stress coefficient is plotted against the side ratio for various conditions of edge restraint and a known compression or tension on the sides of the plate. It is found that there is some reduction in the buckling stress if the sides of the plate are subjected to compressive stresses and, conversely, there is an increase for tensile stresses. Furthermore, for plates with large side ratios, there is a rapid decrease of the spanwise buckling stress as the chordwise compressive stress approaches its appropriate Euler instability value*.

INTRODUCTION

The analysis of rectangular plates under bi-axial compression presented by Wittrick[1] shows that for plates with sides simply-supported it is still possible to use the standard curves for uni-axial compression[2], provided the general formula for the buckling stress is modified to include the compressive or tensile stress on the sides of the plate. For sides built-in, however, the use of uni-axial buckling stress data is not possible. The analysis presented here is basically similar, except that the buckling stress coefficient is given in a form directly applicable for design purposes. In addition, the results for built-in edges on the sides of the plate are also derived.

If a rectangular plate is subjected to a uni-axial compressive stress f_1, it will buckle when f_1 reaches a critical value given by Refs. 2 and 4

$$f_1 = KE(t/b)^2 \quad . \quad . \quad . \quad (1)$$

where K, the buckling stress coefficient, is a function of the side ratio a/b and the conditions of edge restraint. A similar expression can be found for bi-axial compressive stresses f_1 and f_2, except that it is more convenient to introduce the buckling stress as

$$f_1 = kf_0 \quad . \quad . \quad . \quad (2)$$

where $f_0 = \frac{1}{3}\pi^2 E'(t/b)^2$ and k is the buckling stress coefficient which in this case is a function of the edge restraint, the side ratio a/b and the stress ratio f_2/f_0.

NOTATION

x, y	Cartesian co-ordinates, spanwise and chordwise directions respectively
f_1, f_2	direct compressive stresses in the x- and y-directions respectively
E	Young's modulus
$E' = E/(1-\nu^2)$	
ν	Poisson's ratio
K, k	buckling stress coefficients, equations (1) and (2)
$f_0 = \frac{1}{3}\pi^2 E'(t/b)^2$, $(f_0 = 3{\cdot}62E(t/b)^2$ for $\nu = 0{\cdot}3)$	
a	plate length
b	plate width
t	plate thickness
w	lateral deflection
Y	function of y only, equation (4)
m	integer
A, B, C, D	constants of integration
α_1, α_2	parameters defined by equations (7) and (8)

DERIVATION OF BASIC EQUATIONS

Consider a flat rectangular isotropic plate subjected to compressive stresses f_1 and f_2 as shown in Fig. 1, where x and y are directions parallel to the sides and ends of the plate respectively. The ends of the plate are simply-supported while the sides may have various edge conditions. The compressive stresses are taken here as positive and the stress f_2 is assumed to be known; it may be either positive or negative (tension).

The lateral deflection w of a rectangular plate under bi-axial compressive stresses f_1 and f_2 must satisfy the differential equation[3,4]

$$\frac{1}{12}E't^2\left\{\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right\} + f_1\frac{\partial^2 w}{\partial x^2} + f_2\frac{\partial^2 w}{\partial y^2} = 0. \quad . \quad . \quad . \quad . \quad (3)$$

Assuming now that under the action of compressive stresses the plate buckles into m sinusoidal half-waves, the lateral deflection w may be taken as

$$w = Y \sin m\pi x/a \quad . \quad . \quad . \quad (4)$$

where Y is a function of y only. Substituting equation

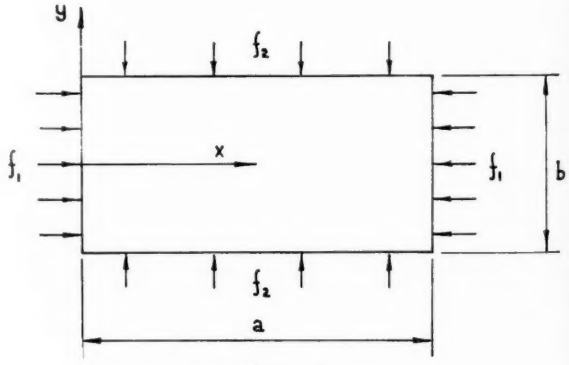


FIGURE 1.

*Since this paper was written the author's attention has been drawn to the fact that results similar to those given here have also been published in an advanced restricted N.A.C.A. report (*see* Ref. 5).

Received 6th May 1955

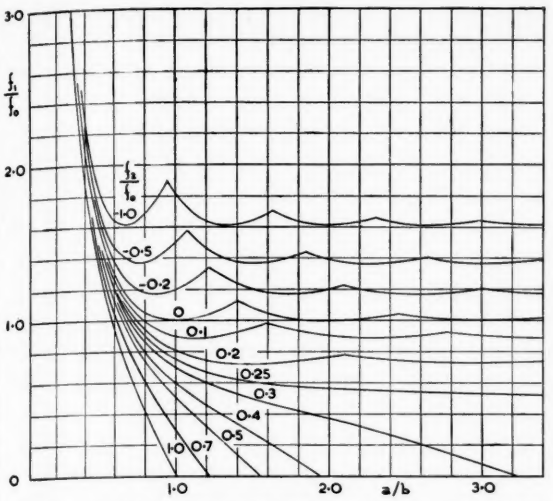


FIGURE 2. Buckling stress coefficient for flat plates under bi-axial compression: ends and sides simply-supported.

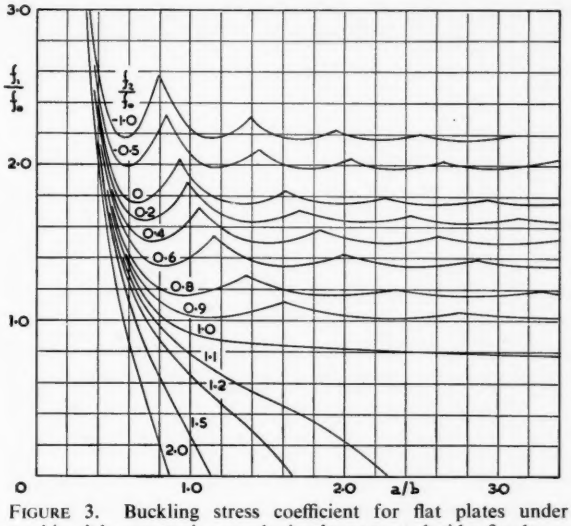


FIGURE 3. Buckling stress coefficient for flat plates under bi-axial compression: ends simply-supported, sides fixed.

(4) into (3) the function Y must satisfy the following differential equation

$$Y_{,4}-2\{(m\pi/a)^2-2(\pi/b)^2(f_2/f_0)\}Y_{,2}+ +(m\pi/a)^2\{(m\pi/a)^2-4(\pi/b)^2(f_1/f_0)\}Y=0 \qquad (5)$$

where $f_0=\frac{1}{3}\pi^2E'(t/b)^2$ and the suffixes denote ordinary derivatives with respect to y. The general solution of this equation is

$$Y=A\cosh\alpha_1\pi y/b+B\cosh\alpha_2\pi y/b+ +C\sinh\alpha_1\pi y/b+D\sinh\alpha_2\pi y/b \quad . \qquad (6)$$

where

$$\alpha_1^2=(mb/a)^2-2(f_2/f_0)+2\{(mb/a)^2(f_1/f_0-f_2/f_0)+ +(f_2/f_0)^2\}^{\frac{1}{2}} \quad . \qquad (7)$$

$$\alpha_2^2=(mb/a)^2-2(f_2/f_0)-2\{(mb/a)^2(f_1/f_0-f_2/f_0)+ +(f_2/f_0)^2\}^{\frac{1}{2}} \quad . \qquad (8)$$

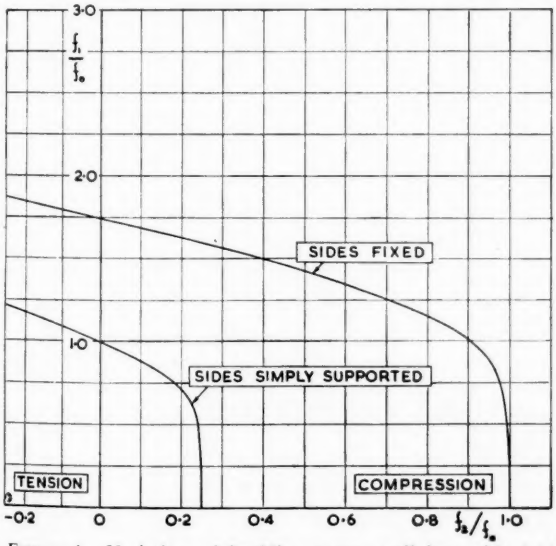


FIGURE 4. Variation of buckling stress coefficient with f_2/f_0 for $(a/b)=\infty$.

and A, B, C, D are the arbitrary constants of integration. The first two terms in equation (6) represent symmetrical deformation about the x-axis and the last two terms represent antisymmetrical deformation.

ENDS AND SIDES SIMPLY-SUPPORTED

Consider first symmetrical deformation about the x-axis so that the deflection w is given by

$$w=(A\cosh\alpha_1\pi y/b+B\cosh\alpha_2\pi y/b)\sin m\pi x/a. \quad . \quad . \quad . \qquad (9)$$

If the sides are simply-supported, then the corresponding boundary conditions at the edges become

$$w=\partial^2w/\partial y^2=0 \qquad \text{at} \qquad y=\pm b/2$$

or

$$Y=Y_{,2}=0 \qquad \text{at} \qquad y=\pm b/2$$

which gives

$$A\cosh\alpha_1\pi/2+B\cosh\alpha_2\pi/2 \quad =0$$
$$\alpha_1^2A\cosh\alpha_1\pi/2+\alpha_2^2B\cosh\alpha_2\pi/2=0.$$

The condition for non-zero values for A and B represents the condition for buckling. This requires that the determinant formed by the coefficients with A and B is identically equal to zero. Hence

$$\begin{vmatrix} \cosh\alpha_1\pi/2 & \cosh\alpha_2\pi/2 \\ \alpha_1^2\cosh\alpha_1\pi/2 & \alpha_2^2\cosh\alpha_2\pi/2 \end{vmatrix}=0$$

or $(\alpha_2^2-\alpha_1^2)\cosh\alpha_1\pi/2\cosh\alpha_2\pi/2=0$ which, in virtue that $\alpha_1^2\neq\alpha_2^2$, becomes $\alpha_1^2=-1$ or $\alpha_2^2=-1$. Either condition reduces finally to the same result giving

$$(f_1/f_0)+(a/mb)^2(f_2/f_0)-\tfrac{1}{4}(a/mb+mb/a)^2=0 \quad . \quad . \quad . \quad . \qquad (10)$$

which determines the buckling stress coefficient $k=f_1/f_0$ in terms of f_2/f_0. The function (10) is represented in Fig. 2 as a system of curves relating f_1/f_0 to a/b for a series of values of f_2/f_0.

In can be shown that the anti-symmetrical deformations produce higher buckling stresses and therefore can be neglected in the present analysis.

ENDS SIMPLY-SUPPORTED, SIDES FIXED

The appropriate boundary conditions in this case are

$$w = \partial w/\partial y = 0 \quad \text{at} \quad y = \pm b/2$$

or

$$Y = Y_1 = 0 \quad \text{at} \quad y = \pm b/2.$$

Considering again only symmetrical modes of buckling, the condition for non-zero values for A and B becomes

$$\begin{vmatrix} \cosh \alpha_1 \pi/2 & \cosh \alpha_2 \pi/2 \\ \alpha_1 \sinh \alpha_1 \pi/2 & \alpha_2 \sinh \alpha_2 \pi/2 \end{vmatrix} = 0.$$

Hence the equation for the buckling stress coefficient f_1/f_0 becomes simply

$$\alpha_1 \tanh \alpha_1 \pi/2 - \alpha_2 \tanh \alpha_2 \pi/2 = 0. \quad . \quad (11)$$

The function (11) is represented in Fig. 3 as a system of curves relating f_1/f_0 to a/b for a series of values of f_2/f_0.

CONCLUSIONS

The curves relating f_1/f_0 to a/b for various values of the chordwise stress parameter f_2/f_0 show that there is some reduction in the buckling stress coefficient for positive (compressive) values of f_2/f_0 and that there is an increase in the buckling coefficient for negative (tensile) values of f_2/f_0.

For convenience the buckling stress coefficient f_1/f_0 is also plotted against the stress parameter f_2/f_0 for $(a/b) = \infty$ (*see* Fig. 4). This shows that there is a rapid decrease in the buckling coefficient as f_2 approaches its appropriate Euler instability stress

$$f_2 = 0{\cdot}25\, f_0 \text{ for sides simply-supported}$$

and

$$f_2 = 1{\cdot}0\ f_0 \text{ for sides fixed.}$$

For a given value of a/b and f_2/f_0 the number of half waves in which a plate buckles along its length is given by a number of troughs to the left of the appropriate side ratio a/b; part of a trough being counted as a complete one.

ACKNOWLEDGMENT

The author wishes to thank The Bristol Aeroplane Company Limited for permission to publish this paper, and Mr. H. L. Cox for his valuable criticism and advice.

REFERENCES

1. WITTRICK, W. H. (1952). Correlation Between Some Stability Problems for Orthotropic and Isotropic Plates under Bi-axial and Uni-axial Direct Stresses. *The Aeronautical Quarterly*, Vol. IV, pp. 83-92, August 1952.
2. Royal Aeronautical Society. Stressed Skin Structures Data Sheets, 02.01.01.
3. COX, H. L. (1954). Computation of Initial Buckling Stress for Sheet-stiffener Combinations. *Journal of the Royal Aeronautical Society*, Vol. 58, No. 525, pp. 634-638, September 1954.
4. TIMOSHENKO, S. (1940). *Theory of Plates and Shells*. First Edition, Chapter VIII. McGraw-Hill Book Co. 1940.
5. LIBOVE, C. and STEIN, M. Charts for Critical Combinations of Longitudinal and Transverse Direct Stress for Flat Rectangular Plates. N.A.C.A. ARR L6A05.

A Chart for Steady Circling Flight

by

Dr-Ing E. SPONDER

(Madras Institute of Technology)

FOR horizontal, or nearly horizontal, steady straight flight there exist some charts from which can be determined the speed at any altitude, the inclination of the flight path, and other quantities. This can be readily done, for instance, by a graphical comparison of the forces of drag and thrust, or by a corresponding representation of the powers, required and available. These charts, however, always involve the polar of the particular aeroplane, and are thus valid only for that aeroplane; they must therefore be replotted for each example. To avoid this and obtain a generally valid chart, the effect of the aerodynamic characteristics of the particular aeroplane must be eliminated; it is thus possible to prepare a chart even for the steady circling flight at any inclination of the flight path.

The three equations for the description of the flight path may be derived on the simple assumption that the thrust acts almost along the direction of flight, *i.e.*, that the angles of attack and of sideslip are negligibly small.

NOTATION

- W weight of the aeroplane (lb.)
- g specific gravitational force (= 32·2 lb./slug at sea level, corresponding to 32·2 ft./sec.²)
- S wing area (or other reference area, ft.²)
- T total thrust (lb.)
- T_0 *see* Fig. 1
- V speed of flight (ft./sec.)
- ρ density of air (slug/ft.³)
- ρ_0 density of air at sea level under Standard conditions $\left(= \frac{1}{420} \text{ slug/ft.}^3 \right)$
- h altitude of flight (thousands of ft.)
- $\sigma = \frac{\rho}{\rho_0}$, relative density of air $\left(\simeq \frac{66 - h}{66 + h} \dagger \text{, valid up to 36,000 ft.} \right)$
- $q = \frac{\rho}{2} V^2$ dynamic pressure of flight (lb./ft.²)
- γ angle of inclination of the flight path, positive in climb (°)
- ϕ angle of bank to right or to left (°)

Received 14th April 1955

†This approximate formula for σ is derived from an essay of Knoller in *Flugsport*, 1924, p. 309, and reproduces the Standard Atmosphere up to the altitude mentioned with less than 1 per cent. error; its inverse is $h \simeq 66\ \frac{1-\sigma}{1+\sigma}$, valid down to $\sigma = 0{\cdot}3$.

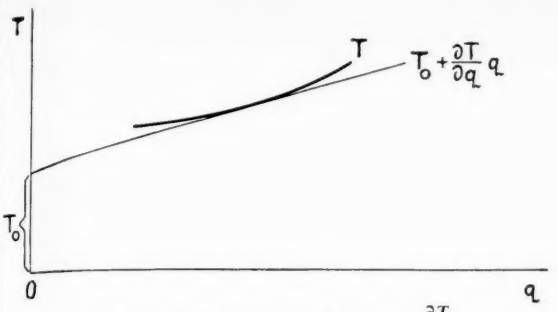


FIGURE 1. Substitution of T by $T_0+\frac{\partial T}{\partial q}q$.

R radius of curvature of the projection of the flight path on the horizontal plane (ft.)

$\Omega=\frac{V\cos\gamma}{R}$, angular velocity of the radius vector R (rad./sec.)

$t_0=\frac{2\pi}{\Omega}$, time required for one complete turn (sec.)

$C_L=\frac{\text{lift}}{Sq}$, lift coefficient of the aeroplane

$C_D=\frac{\text{drag}}{Sq}$, drag coefficient of the aeroplane

$C_T=\frac{\partial T}{\partial q}\Big/S$, slope coefficient of the curve of T against q

$r_a=\frac{C_L}{C_D}$, over-all lift/drag ratio

$r_a'=r_a\Big/\left(1-\frac{C_T}{C_L}r_a\right)$

$n=\frac{C_L Sq}{W}$, load factor

The three reference axes fixed in space, for which is set up the equilibrium of forces in steady circling flight, are:

the direction of flight itself;
a horizontal direction to the right perpendicular to the direction of flight;
the upward direction perpendicular to the two others; the lift makes with it the angle of bank ϕ.

Corresponding to these three directions, the following three equations result

$$C_D Sq=T-W\sin\gamma. \quad . \quad . \quad (1)$$

$$\left.\begin{aligned} C_L Sq\sin\phi&=\frac{W}{g}\,\Omega V\cos\gamma \\ \text{or}\quad &=\frac{W}{gR}(V\cos\gamma)^2, \end{aligned}\right\} \quad . \quad (2)$$

$$C_L Sq\cos\phi=W\cos\gamma. \quad . \quad . \quad . \quad (3)$$

In (1), T is strongly dependent on q; but for the range of speeds of flight in question one can nearly always apply with sufficient accuracy the approximation $T=T_0+(\partial T/\partial q)\,q$ (Fig. 1), in which T_0 as well as $\partial T/\partial q$ are constants.

In addition, introducing $C_T=(\partial T/\partial q)/S$ as the slope coefficient of the curve of T against q, (1) assumes the more convenient form

$$(C_D-C_T)\,Sq=T_0-W\sin\gamma. \quad . \quad . \quad (4)$$

It follows from this that one may replace the variable T with a constant T_0, provided that one diminishes the C_D-values by C_T, therefore, according to the positive or negative sign of C_T, by simply displacing the polar to the left or to the right. An over-all lift/drag ratio $r_a=C_L/C_D$ is then, for the displaced polar, modified to

$$r_a'=\frac{C_L}{C_D-C_T}=r_a\Big/\left(1-\frac{C_T}{C_L}r_a\right).$$

It is assumed for the following that the lift coefficient C_L and the corresponding over-all lift/drag ratio r_a are known, regardless of the conditions which govern their choice, be it the question of the fastest and the sharpest turns* or any other one. It is then the problem to bring the relations (1), (2) and (3) into such a form that they can be applied in a generally valid manner—*i.e.* without being bound to any particular polar—to make a quick analysis of the steady circling flight at that C_L- and r_a-values. For that purpose, dividing (1) by (3):

$$\frac{1}{r_a\cos\phi}=\left(\frac{T}{W}-\sin\gamma\right)\Big/\cos\gamma. \quad . \quad (5)$$

The essential variables here are r_a, ϕ, T/W and γ, the relationship between which is represented in Fig. 2.

Proceeding from the γ-scale at the left, one finds at the inter-section of the T/W-curve in question as a subsidiary result the value of $r_a\cos\phi$ on the scale at the bottom; more important, by moving along the ordinate through this point to the given value of r_a on the right-hand scale, one can interpolate the associated value of ϕ from the family of ϕ-curves. These curves are all parallel, which facilitates the interpolation; besides, they are so arranged that the load factor n can be found with their help, as one has from (3):

$$\frac{C_L Sq}{W}=n=\frac{\cos\gamma}{\cos\phi}. \quad . \quad . \quad (6)$$

This simple relation permits a convenient graphical determination: one need only follow the same ϕ-curve to the intersection with γ and read off then directly under it the value of n, as indicated in Fig. 2 for the subsequent example. For the inverse problem change suitably the marked directions of the arrows.

The only aerodynamic characteristic of the aeroplane involved in this chart is the over-all lift/drag ratio r_a, which of course must be known but may have any

*To determine this, it is necessary to know also the nature of the polar between C_L for $r_{a\,max}$ and $C_{L\,max}$, which region is not covered by the usual parabolic formulae. It is found in many cases that the entire polar of the aeroplane can be represented by a quarter of an ellipse, which can be conveniently transformed into a quarter of a circle by suitably proportioning the scales for C_D and C_L. Given only C_{Do} (at $C_L=0$) and $r_{a\,max}$, and using a judicious estimate of $C_{L\,max}$, it is possible by choosing the ratio of the units for the C_D- and C_L-scales as $\dfrac{2\frac{C_{L\,max}}{C_{Do}}}{\left[\left(\frac{C_{L\,max}}{C_{Do}}\right)\Big/r_{a\,max}\right]^2-1}$, to determine completely this "circular polar," which forms a simple, sufficiently accurate, and reasonable basis for numerical computations (*see*, for instance, Fig. 3).

value. As special cases, both steady straight flight ($\phi = 0$) and gliding flight ($T/W = 0$) are included; its chief purpose is, however, the general treatment of steady circling flight, especially of the fastest and sharpest turns. In this connection it is important to note that if a polar corrected for Mach number is to be used, the corrections should be based on the actual speed of flight round the curve.

As to the scales of the chart, it may be remarked that one must have $r_a \cos\gamma = n\,(r_a \cos\phi) = \text{constant}$; here, the constant is 5, but any suitable value can be chosen.

Now, the speed of flight $V = \sqrt{(2q/\sigma)}$ is determined by equation (6) and may serve for a verification and, if necessary, for a correction of the total thrust T introduced in Fig. 2. Finally, by (1) or (4) too, the result can be checked. The speed of climb during the circling flight is then $V \sin\gamma$.

The questions of the time required for one complete turn t_0 and the radius of curvature R of the projection of the flight path are treated by dividing (2) by (3):

$$\left.\begin{aligned} \tan\phi &= \frac{\Omega V}{g} = \frac{2\pi V}{t_0 g}, \text{ hence } t_0 = \frac{2\pi V}{g\tan\phi} \\ \text{or} \quad &= \frac{V^2\cos\gamma}{Rg}, \text{ hence } R = \frac{V^2\cos\gamma}{g\tan\phi}. \end{aligned}\right\} \quad (7)$$

The best results with regard to the quickness of manoeuvre in circling flight will generally be obtained at over-all lift/drag ratios which are only a little smaller than $r_{a\,max}$. Thus, with the knowledge of $r_{a\,max}$, one would have already a very useful indication of the values of r_a that must be introduced in the chart. It is possible to give average values of $r_{a\,max}$ for the various types of aeroplanes, which may be of good assistance in the absence of better data; Table I* gives the necessary information.

As an example, the performances of a jet fighter which executes its fastest and sharpest turns in climbing

*Chiefly after *Jane's All The World's Aircraft* 1950-1951 and 1952-1953.

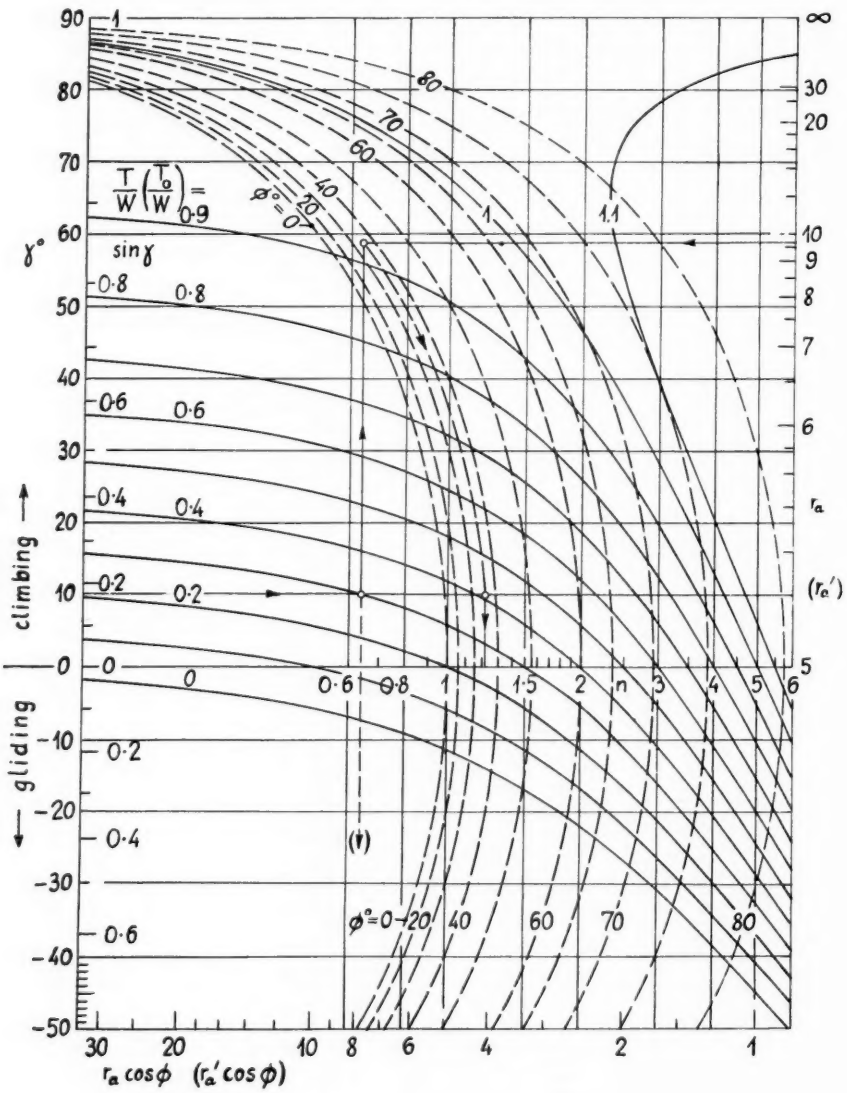


FIGURE 2. Chart for steady circling flight. Independent of all aerodynamic characteristics of the aeroplane, as well as the speed and the altitude of flight.

flight with $\gamma = 10°$ at 13,200 ft. ($\sigma = 2/3$) may now be considered. The constructional data available are:

$W = 11{,}300$ lb., $S = 270$ ft.²;

the power plant is a Rolls-Royce Nene II, 1947.

TABLE I

Type	*Over-all lift/drag ratio* $r_{a\,max}$ *(average value)*
Single-seat fighter	13
Four-engined transport aeroplane	15
Sporting and touring aeroplane	
landing gear retractable	13
landing gear not retractable	11
Sailplane	up to 32†

As no data about the horizontal-flight performance are accessible, the polar can be only guessed at. The C_{D0}-value (at $C_L = 0$) may be about 0·021; a large error here would of course lead to very doubtful results for the maximum speed, but would be of much less importance in the calculations for circling flight. With $r_{a\,max} = 13$ (for single-seat fighters) and a likewise estimated $C_{L\,max} = 1{\cdot}4$, one has all the necessary data for plotting a circular polar as shown in Fig. 3.

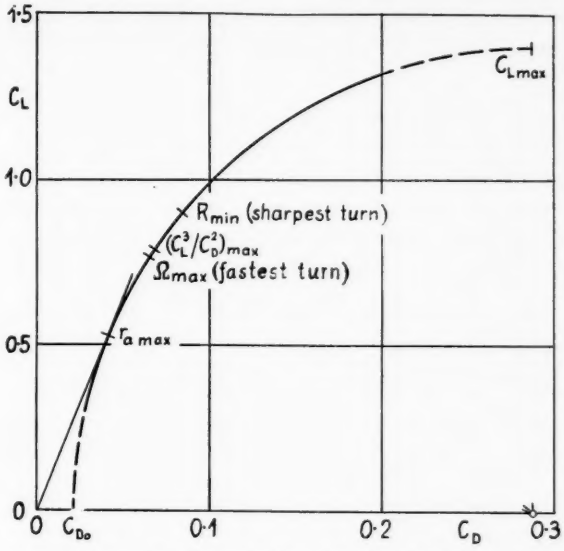


FIGURE 3. The probable polar of the jet fighter.

TABLE II

Flight characteristics	*Fastest turn*	*Sharpest turn*	*Computed from*
C_L	0·77	0·90	any method to find these optimum values
r_a	10·3	9·67	
ϕ	41°	36°	Fig. 2 [or equation (5)]
n	1·30	1·22	
q	70·5 lb./ft.²	56·6 lb./ft.²	equation (6)
V	298 ft./sec. =203 m.p.h.	267 ft./sec. =182 m.p.h.	
$V \sin \gamma$	51·7 ft./sec. =3,100 ft./min.	46·4 ft./sec. =2,780 ft./min.	
R	3,150 ft.	2,980 ft. (minimum)	equations (7)
t_0	67·3 sec. (minimum)	71·1 sec.	

†According to *Aero-Revue* 1951, p. 57.

From the characteristic curves of the jet engine, the relation $T = 3{,}390 - 2{\cdot}7q$ may be taken which covers fairly well the range of the expected speeds of flight.

Hence $T_0 = 3{,}390$ lb., $\dfrac{\partial T}{\partial q} = -2{\cdot}7$ ft.²,

and $\dfrac{\partial T}{\partial q} \Big/ S = C_T = -0{\cdot}010$; $\dfrac{T_0}{W} = 0{\cdot}30$.

The chart for steady circling flight can be used, as indicated by the arrows marked for the sharpest turn in Fig. 2; the results are given in Table II.

It can be seen that the results for the fastest and sharpest turns differ only a little from each other.

ACKOWLEDGMENT

The author is indebted to Mr. Rangarao and Mr. Krishna Iyer, A.R.Ae.S., both of Madras Institute of Technology, India, for the translation.

Effect of Variations in weight or Normal Acceleration on the Ceiling of Jet-Propelled Aircraft — Correspondence

R. J. B. WOODHAMS*

THE TECHNICAL NOTE by Mr. P. L. Sutcliffe in the June JOURNAL (p. 435), apart from repeating well-known proofs that T/σ is constant above the tropopause and for loss of height due to g in turns, seeks to show that rocket boost is necessary for high altitude fighters.

The Note is based on the false assumption that the minimum drag speeds, or speed at $1g$ ceiling, for bomber and fighter are identical, whereas in fact such cannot be the case due to the much higher wing area of the bomber. Further (and from the Note this holds at all altitudes) the fighter must have a high rate of climb at the bomber $1g$ altitude in order to get there in a reasonable time. Normally this well covers the power required for turns.

By and large, one can say that, above the tropopause as the bomber $1g$ ceiling height increases, the ratio of excess thrust required for climb or turns to sea level rated thrust for the fighter decreases. Given the necessary sea level thrust rating to ensure rapid climb to

*Received 27th June 1955.

bomber 1 g ceiling, the fighter design problem would become easier with increasing bomber altitude. The fighter thrust/weight ratio at sea level will of course increase with increasing bomber altitude. This increased thrust/weight ratio of fighter relative to bomber is required at all altitudes in order to produce the necessary minimum time of climb to bomber ceiling.

P. L. SUTCLIFFE

The derivation of the expressions in the Note is independent of the actual minimum drag speed of the aircraft under consideration. The significant relationship is that given in equation (5), and it is worth repeating that this is quite independent of the shape of the aircraft, the thrust of its engines, its maximum altitude at a given value of n and W, or its flight Mach number.

With these facts in mind it is quite possible to compare, for instance, the supersonic fighter and a subsonic bomber on exactly the same basis, the relationship between the various ceilings being taken directly from the graphs given in the Note.

E. N. BRAILSFORD*
(Short Brothers and Harland Ltd.)

Attention should be drawn to the fact that the Technical Note by Mr. P. L. Sutcliffe applies to a particular set of conditions. The purpose of this letter is to emphasise this point because the findings are not general, and therefore are liable to misinterpretation.

The simplifying assumptions in question are that L/D is constant during a manoeuvre or with change of weight, and that initial conditions apply to the absolute 1 g ceiling. These assumptions restrict the aircraft to operating throughout at the lift coefficient for maximum L/D.

In the general case, (4) should read

$$(\phi nW/\sigma)\, M, N, \theta = \text{constant} \quad . \quad . \quad (4a)$$

where $\phi = C_D/C_L$

so that (5) becomes:

$$\frac{n_1 W_1}{n_2 W_2} = \frac{\phi_2}{\phi_1}\frac{\sigma_1}{\sigma_2} = \frac{1}{P}\frac{\sigma_1}{\sigma_2} \quad . \quad . \quad (5a)$$

so that (7) may be written

$$h_1 - h_2 = 20{,}784\left[\log_e \frac{n_2 W_2}{n_1 W_1} - \log_e P\right] \quad . \quad (7a)$$

Using (7a) we see that the correction to (7) is

$$\begin{aligned}\Delta h &= 20{,}784 \log_e P \\ &= 4{,}630 \text{ ft. for } P = 1{\cdot}25 \\ &= 8{,}450 \text{ ft. for } P = 1{\cdot}5 \\ &= 14{,}390 \text{ ft. for } P = 2{\cdot}0. \quad . \quad (8)\end{aligned}$$

The effect of (8) can be shown readily for an interception at constant Mach No., involving a 2 g manoeuvre, where the terminal condition is at the C_L for maximum L/D. In this example, $P = 1{\cdot}25$, so that the maximum reduction in altitude is 14,500 − 4,630 = 9,870 ft. and not 14,500 ft. as might be inferred from Mr. Sutcliffe's Note. (The "steady-state" altitude which follows from this calculation is of course 14,500 ft. below the 1 g absolute ceiling).

Not only are the maximum altitude reductions during interception manoeuvres likely to be less than those implied by (7a), but also these reductions are achieved theoretically only after an infinite time in manoeuvre. The rate of descent is a function of the thrust-drag unbalance which tends to zero at the new "steady-state" altitude. Practical problems concerning interception manoeuvres do not show height losses as large as those obtained from (7a).

Figure 2 refers to the effect of weight change on the absolute ceiling and it will be obvious from the foregoing that it should not be interpreted as applying also to cruising ceilings. The optimum design condition on a long-range aircraft such as a bomber, can be at an L/D away from maximum, even when speed-limited. Weight changes from such a design condition have an effect much smaller than that indicated in Fig. 2.

P. L. SUTCLIFFE

It should be emphasised from the outset that the Note only considers flight in longitudinal equilibrium; climbing and diving flight or flight with longitudinal acceleration are not covered by the equations.

The maximum altitudes, whether 1 g or otherwise, discussed in the Note are related to a given Mach number and are not necessarily the absolute ceilings defined by the maximum lift/drag ratio and, as quoted in reply to Mr. Woodhams, the relationship given in equation (5) is quite independent of the actual minimum drag speed.

The validity of the assumption of constant lift/drag ratio when flying at a given Mach number and engine setting (T/σ), irrespective of altitude, can be shown as follows:—

For the equilibrium condition of thrust equalling drag

$$T = D = C_D \tfrac{1}{2}\rho_0 \sigma S V_T^2$$

or $$T/\sigma = C_D \tfrac{1}{2}\rho_0 S M^2 a^2$$

where a = speed of sound in the stratosphere and the other symbols are defined in the Note.

Now if (T/σ) and M are fixed it is clear that, for any particular aircraft, C_D must be completely specified. This fixes the value of the lift coefficient and consequently the value of the lift/drag ratio, irrespective of the altitude.

This being so, Mr. Brailsford's equation reduces to that of the Note, *i.e.*

$$\left(\phi, \frac{nW}{\sigma}\right)_{M,N,\theta} \equiv \left(\frac{nW}{\sigma}\right)_{M,N,\theta,\phi} = \text{constant}$$

and in his terminology $P = 1{\cdot}0$, $\text{Log}_e P = 0$, $\Delta h = 0$. It should be noted that values of P other than unity indicate violation of the conditions of M and (T/σ) constant and steady level flight.

Furthermore, in the case of the long range aircraft, the optimum flight plan for either range or altitude requires both the engine setting and the Mach number (and inherently the lift/drag ratio) to be fixed. Thus the effect of weight changes during a ceiling cruise are correctly given by the equations of the Note.

Received 7th July 1955.

THE LIBRARY
Reviews

GAS TURBINES FOR AIRCRAFT. *Ivan H. Driggs and Otis E. Lancaster. Ronald Press Co.: New York. 349 pp. Illustrated. $10.00.*

This book, by the Chief Scientist, U.S. Naval Development Centre, and an Assistant Director, Bureau of Aeronautics, raises expectations which are, on balance, well realised. The inclusive title is perhaps rather misleading, for until the reader realises that the main purpose of the early chapters is to provide background and introduction to Chapters 8, 9 and 10, he may feel some parts of them to be sketchy and inadequate. The main justification for the book lies in fact in these later chapters, which deal with Gas Turbine Performance, Control Problems, and Aircraft Performance Analysis. In these chapters one feels that the authors are on their own ground, and it is on them that the book should be judged. They might well have been amplified, even at the expense of more cursory treatment earlier. They would justify, in the reviewer's opinion, the more suitable title of " Aircraft Gas Turbine Performance."

The earlier chapters deal with Perspective (a useful historical survey), Fundamentals, Cycle Analysis, Compressible Flow, Compressors, Turbines, and Combustion, and are on the whole well done. The authors' treatment is fundamental, and directed to the " why " rather than the " how." The reviewer has no quarrel with this in principle, but confesses (is this insular prejudice?) to a preference for evaluating cycle efficiencies and output on the "traditional" basis of maximum temperature, component efficiencies, and mean specific heat, rather than through changes in enthalpy and entropy. Such a treatment may lend itself to the rapid evaluation of ideal cycles (using the charts given in the book) but it is apt to overlook the essential cycle limitations with which the designer is concerned.

The description of flow phenomenon in diffusers and nozzles is lucid and worthwhile. The principles of centrifugal and axial compressors are discussed briefly, although in view of the extensive use which is made later of " dimensionless " performance parameters, it is surprising that this method of plotting should receive little more than passing reference. The principles of turbines are covered in ten pages, after which one feels that the authors turn with relief to a review of blade and disc materials and cooling methods. Chapter 7 on Combustion is rather superficial; the terms " combustion chamber," " combustor " and " burner " are used variously and interchangeably. However, in view of the consistent use of enthalpy rather than temperature throughout the book, it is refreshing to find here the admission that: " Unfortunately the strength of the metal parts responds to temperatures rather than enthalpies "!

Chapters 8, 9 and 10 are really worth while and repay careful reading. One might only have asked for more detailed treatment. (For example, the discussion of what we should call equilibrium running conditions for the turbo-jet is rather over-simplified by the assumption of an infinitely-variable propelling nozzle, and the uniqueness of the engine running line (with a choked fixed nozzle) thereby missed.) The basic problems of turbo-jet control are well illustrated by a number of proposed operational schedules, and acceleration limits are clearly if briefly discussed. Methods of estimating engine weight are described and formulae derived for total engine and fuel weight for various operating conditions. In Chapter 10 the authors describe a novel and rapid method devised by them (and used by the Bureau of Aeronautics) for predicting the complete aircraft performance on the basis of generalised " thrust required " curves. The advanced student will undoubtedly find these chapters stimulating and valuable. The book is admirably printed, illustrated and produced.—E. GLAISTER.

FLUID DYNAMICS OF JETS. *Shih-I Pai. D. Van Nostrand Co. Inc., New York, 1954. 227 pp. Illustrated. 37s. 6d.*

The first impression given by the title of this book is that jet flow cannot be satisfactorily treated in isolation, as it is only a particular section of general fluid dynamics. If this view is correct it follows that many parts of the general subject, for example, conformal transformation, have to be cursorily described in order to explain the application of the general theory to the flow of a jet. The author of the present book has failed to resolve this difficulty, because in fact no solution is possible within the chosen framework. Nevertheless Dr. Pai has written a valuable book which collects a lot of information on recent theoretical investigations of jet flow; experimental results are given in some cases where a comparison with theory is possible, but the emphasis is on the theories.

The first three chapters deal with inviscid flow at subsonic and supersonic speeds: this ground has been adequately covered in general works on fluid mechanics. The next four chapters deal with laminar and turbulent jet flow, including the case of the mixing of two streams of different velocities. Wakes which, as the author points out, are similar in many respects to jets, are, however, not discussed in these chapters. The temperature distribution in heated jets is not thoroughly discussed and this omission is regrettable. An account of the mixing of streams and jets of different gases is included and this has not been presented previously.

The last two chapters deal with the stability of jet flows of incompressible and of compressible fluid. The author has done research on this subject himself, and he has given an up-to-date, though unavoidably incomplete, account of it. It would have been useful if he had indicated the conditions under which laminar jets and stability are important: any jet which is not fully turbulent is, in fact, a rare occurrence.

Dr. Pai writes clearly and has given us a useful book, but it is a book which will be mainly used as a reference work and is not suitable for use as a text book.—H.B.S.

AIRPLANES OF THE WORLD. *Douglas Rolfe and Alexis Dawydoff. Simon and Schuster Inc., New York, 1954. 320 pp. Illustrated. $2.95.*

In five years time, when my small son is old enough to view an Avro 504 as one of the great aeroplanes of history, instead of as an old-fashioned stick-and-string monstrosity, this is the book I shall take from the shelf for an evening's reminiscing.

It contains drawings by Douglas Rolfe of 843 aircraft spanning five centuries, from Leonardo da Vinci's man-powered ornithopter to the delta-wing Avro Vulcan and supersonic research aircraft of today. The selection is first-class, covering almost all of the aeroplanes one looks for in a historical review, and quite a lot of the lesser known, but no less interesting, experimental types.

Most notable omissions are Ellehammer's pioneer aircraft of the 1904-7 era, Fabre's seaplane of 1910, the Heinkel He 178, and Russian types such as Mozhaisky's 1882 monoplane—claimed by Moscow as the first powered aeroplane to fly—and the modern MIG-15 and IL-28.

The style of the drawings varies considerably, giving the impression that they were produced over a period of several years for different magazines, and then gathered together for publication in this book. A few of them will offend the enthusiasts who insist on perfect technical accuracy; but, particularly in the pre-1914 era, they display the aircrafts' " character " more clearly than would a faded or heavily-retouched photograph.

Each of the nine sections of the book is introduced by a brief review of aviation development in the period covered by the drawings that follow. Written by Alexis Dawydoff, this narrative is readable and far more comprehensive than one would expect in a total of only 10,000 words. It contains some rather remarkable Americanisms, and a number of mis-spellings and factual errors. For example, Sperry invented the first automatic pilot in 1912, not 1932; the R.F.C. was established in 1912, not 1911; the designer of the Antoinette was Levavasseur, not Levasseur; the " Bloater " was the B.E.8, not B.E.3; the Pfalz D.III was more, not less, rugged than the Albatros D.V; and the three-seater of 1913 was not the first Sopwith tractor design.

These are mere pin-pricks in a book that will enable many an old-timer to relax in an armchair, turn its pages slowly and recall the days when flying really *was* flying.—JOHN W. R. TAYLOR.

STRÖMUNG DURCH ROHRE; UMSTRÖMUNG VON KROPEN BEI ZWEIDIMENSIONALER STRÖMUNG; UMSTRÖMUNG VON KORPEN BEI RAUMLICHER STRÖMUNG. (*Flow through tubes; flow past two-dimensional bodies; flow past three-dimensional bodies.*) *L. Schiller. Springer-Verlag, Berlin. 134 pp. Diagrams. In German.*

This is a reprint of a section of the first part of Vol. I of the Randolt-Bornstein Numerical Values and Functions for Physics, Chemistry, Astronomy, Geophysics and Technology. Tables and curves together with numerous references are given for the aerodynamic properties of pipes and bends, radiators, flat plates, surface excrescences, aerofoils and wings, bodies of revolution, undercarriages and the numerous bits and pieces associated with aircraft. This volume forms a useful companion to the well known work of Hoerner (" Aerodynamic Drag ") although it is to be regretted that so much data is given with so little discussion. —J.A.D.

JAHRBUCH 1953 DER WISSENSCHAFTLICHEN GESELLSCHAFT FUR LUFTFAHRT. *Edited by H. Blenk. Vieweg and Sohn, Braunschweig, 1954. 224 pp. Illustrated. DM.28. In German.*

Previous volumes of Jahrbücher in the library cover the years 1921-1926 and 1929 and while the style may have changed, the excellence of the papers is still what one has learnt to expect from Germany. The present Year Book covers the annual conference of W.G.L. held at Göttingen in 1953 and is a collection of authoritative papers on all aspects of aeronautics, but mostly on aerodynamics; one is able to gauge their quality by such well-known names as Professor Young and Dr. Lachmann. There is a summary of each paper in French and English.

A short while ago a daily paper's air correspondent expressed his disbelief in the possibility of a renaissance of German aeronautics in, I think, 20 years. He should glance at this annual.—F.H.S.

THE OBSERVER'S BOOK OF WEATHER. *Reginald M. Lester. Frederick Warne, 1955. 151 pp. Illustrated. 5s.*

This small book gives an interesting description of the main weather phenomena in terms which can be readily understood by anyone who is interested in the subject. It is not, neither is it intended to be, a book for the expert, although even he may find here a few simple rules to enable him to fix more accurately the dates for his picnics. The colour plates add to the beauty of the volume but are not always particularly clear.—D.C.S

THE CONTINUUM AND OTHER TYPES OF SERIAL ORDER. *E. V. Huntington. Second edition. 1955. 80 pp. 1 dollar.*

ALMOST PERIODIC FUNCTIONS. *A. S. Besicovitch. 1954. 180 pp. 1 dollar 75.*

TREATISE ON THERMODYNAMICS. *Max Planck (Translated by Alexander Ogg) Third Ed. 1955. 297 pp. 1 dollar 75.*

ADVANCED DYNAMICS OF A SYSTEM OF RIGID BODIES. *E. J. Routh. Sixth Ed. 1955. 484 pp. 1 dollar 95.*

THE DYNAMICAL THEORY OF GASES. *Sir James Jeans. Fourth Ed. Dover Publications, New York, 1955. 444 pp. 2 dollars.*

The publication dates given are of the special reprints that Dover Publications have prepared of these old classics. The first was originally produced by Harvard in 1917 and the second by C.U.P. in 1932, and Dover Publications are performing a great service by reprinting these titles which students must have spent much time seeking in secondhand bookshops. None of the books needs reviewing in the usual sense because each is already well known. It is hoped that this firm will continue the good work and this reviewer would like to suggest Todhunter and Pearson " History of the Theory of Elasticity " as a future title.—F.H.S.

MANUFACTURE AND APPLICATION OF LUBRICATING GREASES. *C. J. Boner. Reinhold Publishing Corporation, New York; Chapman & Hall Ltd., London, 1954. 977 pp. Illustrated. 148s.*

During the past fifteen years or so, a vast amount of effort has been directed to the study of methods of manufacture and application of lubricating greases. This has resulted in grease technology, in all its aspects, being placed on a more scientific basis and becoming more intelligible to the workers in the field. There was, however, an urgent need for a reference book on the subject and the present volume amply fulfils this requirement. A very extensive survey of all available results has been made by the author (himself a prominent worker in the field) and an exhaustive bibliography is included.

The book is stated to be for the use of " manufacturers, research chemists, lubricating engineers, raw materials suppliers and all users of grease lubricants." It will probably be found that most benefit will be gained by workers in the productive side of the industry, since this aspect is very well covered. The chapter on raw materials is very compact, yet provides the essential information. Manufacturing processes are dealt with for all the common soap thickeners, including " mixed " and " complex " soaps, with comprehensive details of equipment used, and the latest types of non-soap thickeners are discussed.

The book will prove to be valuable also to research workers, particularly the new entrants into the field, who are now provided with an up-to-date picture of most of the

significant developments. For lubricating engineers and general users of greases, it is felt that the volume will prove to be somewhat less informative. Although details are provided of most of the routine tests, the significance of the latter is not always stressed sufficiently and the chapter on application of lubricating greases is more or less restricted to methods of dispensing.

The author recognises that future developments in all aspects of the grease industry will depend largely upon further advances in our knowledge of grease structure. The section of this book dealing with structure and theory, while it covers the published work, deals largely with actual fibre structure, as determined by X-ray and electron microscope studies. The results, however, provide an incomplete picture and for this reason the succeeding sections of the book are still somewhat empirical. More recent research involving studies of the state of aggregation of the soap fibres is producing a much clearer picture of the effect of processing upon grease structure and, equally important, the effect of structure upon performance.—D. EVANS.

LECTURES ON PARTIAL DIFFERENTIAL EQUATIONS. *I. G. Petrovsky. Translated from the Russian by A. Shenitzer. Interscience Publishers, New York, 1954. 245 pp. 41s.*

The book contains supplemented notes of lectures given by Professor Petrovsky to students of mathematics in the Department of Mathematics and Mechanics of the Moscow State University.

The first thing about the book which impresses itself upon the reviewer is the excellent way in which the author has included both rigorous proof and physical explanation. For example, as early as page 2 the author discusses in clear physical terms the derivation of the equation of heat conduction. He then discusses physically what boundary and initial conditions would suffice to determine completely the solution to the problem, distinguishing clearly between the different types of possible boundary and initial conditions. Similarly the author considers vibrations of a membrane. This is immediately followed by rigorous considerations of the conditions under which the Cauchy problem has a unique analytic solution. This happy marriage, achieved so early in the book, continues to the end, making it both readable and complete.

The book is divided clearly and conveniently into four chapters. Chapter 1 deals with the classification of equations. The conditions for the existence of a unique solution are discussed rigorously, and it is shown how various types of equation may be reduced to canonical form.

Chapter 2 deals with hyperbolic equations, and contains such widely varied physical applications as (*i*) Kirchhoff's formula for the solution of the wave equation (*ii*) the Lorentz transformation and special relativity (*iii*) vibrations of strings and membranes. It also includes such important mathematical concepts as (*a*) the "Fourier" method of separation of variables and (*b*) general properties of eigenfunctions and eigenvalues.

Chapter 3 considers elliptic equations, and begins by defining the Dirichlet problem for a circle and solving in the form of Poisson's integral. The author examines with great vigour the existence of a solution of the general Dirichlet problem, and discusses how it may be solved approximately by the method of finite differences. Physical applications deal with potential theory.

Chapter 4 deals briefly with parabolic equations, with application to the conduction of heat.

Although apparently written primarily for graduate students it will be valuable also to undergraduate students of applied mathematics, provided they are willing to pass over a few of the more difficult passages and concentrate mainly upon the applications to specific physical problems. The reviewer highly recommends the book to all who wish to further their understanding of the solution of partial differential equations.—N. CURLE.

ANALYSIS OF DEFORMATION. *Vol. II. K. Swainger. Chapman and Hall, London, 1954. 365 pp. Diagrams. 70s.*

In Volume I of this work Mr. Swainger formulated a linear theory to analyse the finite deformation of all amorphous continuous substances. General inferences were drawn without solving equations and boundary conditions for particular cases. Volume II is now concerned with the application of the theory and with the examination of such experimental evidence as is available for the equilibrium deformation of various solids having various boundary conditions.

That there is at present little experimental evidence on the behaviour of engineering metals under complex stress is a point much laboured in the author's preface. Some of the tests undertaken by the author to add to the experimental evidence were apparently the cause of some controversy in the case of values of the plastic transverse contraction ratio. Whatever the rights and wrongs of the controversy there remains the point that the experimental evidence is, as yet, sparse and there is a school of thought in engineering which would not venture very far with the application of the theory until the experimental corroboration was more complete.

Beginning with the consideration of one-stress definitions and tests the author proceeds through the various stress conditions giving theoretical examples and experimental results where appropriate. There are special chapters on rubber deformation and its theoretical interpretation, couples applied to massive bodies and the shearing of thin plane laminae.

Vector analysis, complex variables, finite differences and Fourier series are conveniently covered in Appendices for a better understanding of the mathematics involved in the author's treatise.—A.J.B.

THE FIRST AND THE LAST. *Adolf Galland. Methuen & Co. Ltd., 1955. 368 pp. Illustrated. 18s.*

Books about the last War, as seen from the German side, have been appearing in increasing numbers translated into English in the last year or two. Of those dealing with the War in the Air, "The First and the Last" by Lieutenant General Adolf Galland, the outstanding and much-decorated German fighter pilot who was the Luftwaffe's "General of the Fighters Arm" from 1941 to 1945, is undoubtedly outstanding.

It is of interest not only as a dramatic record of an operational pilot's experiences but also as an exceptionally well-informed account of the higher policies of the Luftwaffe as they were known to a key senior officer who had unusual qualifications to see and interpret the problems and decisions of the High Command while retaining his own contacts with operational flying. As a result of the vicissitudes of high office in Germany under the National Socialist régime, Galland—who had started the War as a Lieutenant in command of a fighter squadron—finished the War again leading a fighter squadron into action, in spite of the fact that he was by then a Lieutenant General! This first-hand experience of the Luftwaffe's actual job both at the first and the last stages of the War enables him to see unusually clearly the operational significance of the many striking technical advances in the design and

manufacture of aircraft and equipment with which the Germans sought to change the tide of war in its later stages.

This book has claims to historical importance as a personal record of the Lutwaffe during the Second World War. But, much more than this, it is of value as an interpretation of the effect of technical advance on the technique of air warfare as it was practised during those years by the German Air Force. Not everybody will agree with all Galland's assessments and conclusions but it is clear that he states a German point of view which, if it does not always agree with the contemporary official German viewpoint—indeed, is often strongly critical of it—is yet one which is well-informed and which, if it had prevailed at the time, might have had far-reaching effects on the progress, perhaps even on the outcome, of the War.

The reader finds, however, that there are not a great many facts in this book which interested people do not already know. Indeed, in one or two places Galland himself appears to have got his facts wrong. Undoubtedly of most value is his outline of German aircraft development during those fateful years. He relates this to the contemporary political and operational scene and to the practical experiences of an outstanding pilot so as to give old facts a fresh and fascinating new perspective. The story is told straightforwardly, without undue recrimination or excuses, in a way which makes it absorbingly interesting.—P. W. BROOKS.

SPONTANEOUS IGNITION OF LIQUID FLUIDS. *B. P. Mullins. Butterworths Scientific Publications, London, 1955. 117 pp. Diagrams. 20s.*

This monograph (Agardograph No. 4) was written at the request of the Combustion Panel Advisory Group for Aeronautical Research and Development within N.A.T.O. and its publication has been sponsored by this group. The author, who is chairman of the Combustion Panel, is Head of the Chemical Physics Department, National Gas Turbine Establishment of the Ministry of Supply and has for many years been engaged in the study of combustion processes. In his preface the author states that the book " is essentially a review and assessment of the state of knowledge in one area of the combustion field, but it should serve in addition as a source book of the great amount of experimental information on spontaneous ignition published during the present century." Unfortunately, perhaps, the author has adhered rigidly to the terms of his assignment and presented a comprehensive review of the literature on methods of determination of spontaneous ignition temperatures with only tantalisingly brief reference to the applications of this knowledge to such problems as static fire hazards, spark ignition and compression-ignition engines.

The book contains fifteen chapters, the first two dealing with General Theoretical Considerations, the next eight with the determination of spontaneous ignition temperature by the Moore, Jentzsch, Furnace, Bomb, Adiabatic Compression, Flow, Heated Surface and miscellaneous methods. Chapter eleven consists of data on spontaneous ignition of 433 substances which might have been more conveniently presented in tabular form, but which nevertheless represents a most important collection of data. The remaining chapters deal respectively with the effect of fuel additives, factors affecting ignition delay, applications and conclusions.

The monograph is obviously intended for the specialist, for research workers in the fields of combustion, fuel properties and engine design, but it will be of considerable interest to other workers in related fields. For this reason it might have been desirable for the author to have expanded the text a little in certain respects, particularly in the chapter on general theoretical considerations and in the chapter on applications. This is, however, a minor criticism of a most excellent monograph which within its one hundred pages contains a critical and at all times constructive review of some 1,350 scientific papers. Perhaps the most interesting parts of the text are those in which the author allows himself to depart a little from his position as a reviewer and points out the areas within the subject concerning which further knowledge is urgently needed. While serving as reference text to all workers in the field of combustion, the book will also direct and inspire new researches in this field.

The book is well produced and illustrated with excellent line drawings. Despite the volume of work reviewed the text is eminently readable and well indexed. The price is obviously below cost and the book should find a place in all fuel and engine laboratories.—F. MORTON.

HIGH SPEED AERODYNAMICS AND JET PROPULSION. *Volume VI. General Theory of High Speed Aerodynamics. Edited by W. R. Sears. Oxford University Press, 1955. 758 pp. Diagrams. 100s.*

This is the first volume to be published of a projected twelve volume series, usually referred to as the Princeton Volumes and having the general title " High Speed Aerodynamics and Jet Propulsion." From the point of view of the theoretical worker, no volume of the twelve could have been more welcome as a curtain raiser than this volume with its title " General Theory of High Speed Aerodynamics." The editor, Professor W. R. Sears, and his very distinguished list of contributors, are to be congratulated on producing a very stimulating work.

Books of this nature which are so fascinating to read present the reviewer with serious problems. It is difficult within the space allotted to a review to give some idea of the general scope and content and at the same time to avoid sweeping generalisations about the general presentation and outlook which, let it be said at once, are no more uniform here than in other comparable composite works.

The volume opens with an introduction by Professor von Kármán. What more could an editor dream of than an introduction by Kármán, and here it is fulfilling his dreams and mine. This is followed by a section on the Mathematical Aspect of Flow Problems of Hyperbole Type by K. O. Friedrichs which is short, to the point and very well written. The editor himself has written the section on " Small Perturbation Theory " and has produced the results one would expect of him when batting on his own ground. He has collaborated with Y. H. Quo in the section on " Plane Subsonic and Transonic Potential Flows." The longest section, some 218 pages, is devoted to " Supersonic and Transonic Small Perturbation Theory " and has been written by M. A. Heaslet and H. Lomax. It is almost a book in itself and could, I feel, have been materially shortened without serious loss, indeed with some benefit to the reader. The section on " Higher Approximations " has been written by M. J. Lighthill. The editor singles it out in his preface and describes it as a piece of work unique in aeronautical literature and I would certainly not cavil at this description. The last two sections, one on the " Methods of Characteristics " and the other on " Supersonic Flows and Shock Waves " have been contributed by A. Ferri. I formed the impression that in the former the wealth of detail offered, much of which could reasonably be expected to be supplied by the reader, made the subject as well as the section much more cumbrous than it need be.

It will be clear from this summary that the theory developed is entirely inviscid. Viscous effects and turbulence are reserved for Volumes 4 and 5.

The type of reader for whom the work is written is indicated by the remark in the preface that "The need for a treatment of this type has long been felt in research institutions, universities, and private industry and its potential reflected importance in the advanced training of nascent aeronautical scientists has also been an important motivation in this undertaking." Certainly it is clear that a sound mathematical background is an essential prerequisite. The state of the subject is such that I for one would have taken exception to any other basis for a volume such as the one under review, though it has not escaped criticism on this account elsewhere.

To one who has had to try to condense the material of about five of these volumes into the space occupied by one, the space available seems incredibly lavish. Whether any two people would agree precisely with the editor's choice of how the space should be filled is exceedingly doubtful. It is difficult to avoid a feeling in places that the policy has been to insure against omissions by including everything possible. For my part I would have welcomed a closer sifting of the material with an eye on what is likely to stand the test of time, even at the risk of some omissions. However, this may be put down as carping criticism or even jealousy from one whose problem was so different.

One of the advantages of the spaciousness is that it has been possible to say something about problems unsolved at the time of writing. For the research worker this adds a spice to the pleasures of reading this volume and does something to offset the earlier criticisms of excessive length. Perhaps I could illustrate by reference to the section by Sears and Kuo on "Plane Subsonic and Transonic Potential Flows." They discuss the breakdown of the hodograph method and the possibility of smooth (shock-free) transonic flows. Special solutions showing such flows have been obtained theoretically, but experimentally shock waves always seem to occur and Busemann has given a possible explanation. Sears and Kuo conclude by saying, "The argument presented by Busemann is surely plausible from the physical standpoint. Nevertheless his conjecture has not been convincingly proved in a mathematical way, possibly because the formulation of the problem has been inadequate." What nicer way is there of throwing down the gauntlet?

Other examples may not be so clearly ticketed but they are there for all to see none the less. Ferri's last section on "Supersonic Flows with Shock Waves," for example, is a very happy hunting ground. Some discussions such as the flow past a yawed cone and in particular the associated entropy field are clearly incomplete. Others such as the location of the sonic point in transonic flow past boundaries with sudden changes of slope raise, at least in my opinion, as many problems as the treatment given answers.

The book contains in addition a wealth of references, many cited at the appropriate place in the text, others additional to the material of the text. They form not the least valuable part of the book.

In short, the book is a welcome addition to the library of aeronautical works. It will find, expensive though it is, a place on the shelves of those whose spirit is stirred either by the present-day achievements of the theory of this science or by the promise of things to come.—L. HOWARTH.

FRONTIER TO SPACE. *Eric Burgess. Chapman and Hall, London, 1955. 174 pp. Illustrated. 21s.*

The recent announcement that Government support is to be given for a British programme of high-altitude rocket research (at least up to the modest height of 60 miles) makes the appearance of this book very topical. Its title might suggest that it is yet another of the many current books of a popular nature on the possibilities of space flight, but actually it is a quite comprehensive and fairly technical review of achievements to date in the use of rockets for various kinds of physical research in the upper atmosphere. As such, of course, its name is none the less apt.

There is one chapter largely devoted to the history and particulars of the rocket vehicles themselves (Aerobee, V.2, Viking, and so on) and another which discusses at some length the future possibilities of more ambitious unmanned missiles as artificial satellites, and as expendable "probes" for one-way interplanetary voyages, for example, to a reconnaissance orbit around Mars. In each case, scientific data would be telemetered back to the terrestrial base.

For the rest, however, apart from a considerable number of very interesting illustrations, the material of the book is likely to be of greater value to physicists concerned with its main theme than to rocket engineers, for whom it is not principally intended.

Present knowledge on the structure and phenomena of the atmosphere is surveyed and summarised. Cosmic and solar radiation, the propagation of radio waves as affected by the ionised layers at great heights, meteors, and other subjects of a similarly relevant nature are dealt with. So also are the questions of the instrumentation and techniques of telemetry employed in the research rockets which have been used to investigate these subjects.

All these matters have previously been dealt with at greater length, by the specialists concerned, in Newell's "High-Altitude Rocket Research," and in "Rocket Research in the Upper Atmosphere"—the proceedings of the 1953 Oxford Conference on this subject, edited by Prof. Massey. Both books are quoted as references in the work now under review, which features excellent bibliographies at each chapter ending and for the book as a whole.

Mr. Burgess's volume might perhaps best be recommended to those having a general or fringe interest in upper atmospheric phenomena, or to those just entering this field of study, as an introduction to the authoritative sources mentioned above. Viewed in this light, it is felt that it can be given a very definite recommendation, as the most complete review available at a reasonable price, within the knowledge of the present writer.—A. V. CLEAVER.

WHO'S WHO IN WORLD AVIATION. *Edited by Wayne Parrish. American Aviation Publications Inc., 1955. 345 pp. $10.*

It is a pity that the price of this book is so high but one must bear in mind that it has taken two years to compile.

Only time will prove the extent of its usefulness and find its omissions but 2,400 names must include most of the notabilities in aeronautics. One is, perhaps, a little surprised to find that the combined Honorary Fellowships of the Society and the Institute are not enough to qualify the former Secretary of the Society. The reparation of such omissions, where they occur, is promised in the next edition.

A straightforward alphabetical arrangement (those perennial Mac people coming after Maz for a change)

makes the book easy to use and the biographical notes to most names are sufficiently detailed. All branches of aviation are catered for, government officials, pilot organisations, air force officers, civil executives and—as the title says—the " coverage " is World-wide.—F.R.S.

THE AIRPORT VISITOR AND AIR TRAVELLER. *Compiled by G. A. Frais. Penman Enterprises Ltd., 1955. 72 pp. Illustrated. 2s.*

A new edition of this booklet has just appeared, and it has several new features. A " Why and Wherefore " section for air travellers, including sample lists of the amount of clothing which can be carried within the 44 lb. and 66 lb. luggage allowance; a short article on interplanetary travel; and an " Aircraft Check Chart " in the form of a family tree, with civil aircraft classified according to their landing gear, and the number of their engines and fins. The descriptions of aircraft are fuller than before, and are classified in the same way, which should make identification simpler for the interested airport visitor.

FLEXURE DEVICES—PIVOTS, MOVEMENTS, SUSPENSIONS. *P. J. Geary. Published by the British Scientific Instrument Research Association. 44 pp. Diagrams. 10s. 6d.*

This is the first of a series of bibliographical surveys on instruments parts. It comprises a 13 pp. technical introduction with 21 diagrams, digests of 28 selected texts of major importance, and annotated lists of 128 other texts of lesser or limited importance, together with prefatory matter and indexes. The period covered is from 1900 to date. The subject matter includes Flexure Pivots (*e.g.* cross-spring pivots); Parallel Movements; Twin Strip Mechanisms; Buckling Strip Devices; The Sine Spring.

The technical introduction makes clear the advantages of these devices (*e.g.* absence of friction, backlash, hysteresis, wear and need for lubrication) and indicates their limitations.

CONTINENTAL MILITARY AIRCRAFT. *J. W. R. Taylor. 75 pp. Illustrated.*

BRITISH MILITARY AIRCRAFT. *J. W. R. Taylor. 75 pp. Illustrated. Ian Allan Ltd., London. 2s. 6d. each.*

The Ian Allan abc series really need no review apart from an announcement that a new title has appeared. Anyone who has browsed at a Smith's bookstall knows their make-up and the scope of each is explained in the title. Nevertheless, I could wish for a more satisfactory layout. Even with the explanation given in the Introduction I found it most difficult to follow the alphabetical order. And even though it is in service with the Royal Navy I must say I resent the presence of a Douglas Skyraider in a book of British Military Aircraft.—F.H.S.

Additions to the Library

Biot, M. A. THE DIVERGENCE OF SUPERSONIC WINGS INCLUDING CHORDWISE BENDING. (C.A.L. Report 67.) Cornell Aeronautical Laboratory. 1954.

Blough, G. O. MASTERS OF THE AIR. Smithsonian. 1954.

Brombacher, W. G. and T. W. Lashof. BIBLIOGRAPHY AND INDEX ON DYNAMIC PRESSURE MEASUREMENT. U.S.G.P.O. 1955.

Bureau of Medicine and Surgery. AVIATION MEDICINE PRACTICE. U.S.G.P.O. 1955.

Chapel, C. H. *et al.* AIRCRAFT POWER PLANTS. McGraw-Hill. 1955.

Goodman, T. R. A THEORY OF SHOCK WAVE BOUNDARY LAYER INTERACTION. Cornell Aeronautical Laboratory. 1953.

Grover, H. J. *et al.* THE FATIGUE OF METALS AND STRUCTURES. NAVAER 00-25-534. U.S.G.P.O. 1954.

Jeans, J. H. THE DYNAMICAL THEORY OF GASES (4th edition). Dover 1955. (C.U.P. 1925.)

Latimer-Needham, C. H. THE FLIGHT REFUELLED BOMBER. Flight Refuelling. 1955.

Mahlmann, C. V. and W. M. Murray (*Editors*). PROCEEDINGS: THE SOCIETY FOR EXPERIMENTAL STRESS ANALYSIS. Vol. XII, No. 2. S.E.S.A. 1955.

Marquard, E. VIBRATION DYNAMICS OF THE FAST ROAD-VEHICLE (UNDERCARRIAGE APPLICATIONS). Cornell Aeronautical Laboratory. 1954.

N.P.L. NOISE MEASUREMENT TECHNIQUES. (NOTES ON APPLIED SCIENCE No. 10.) H.M.S.O. 1955.

Planck, M. TREATISE ON THERMODYNAMICS (3rd edition). Dover 1955. (Longmans Green 1922.)

Routh, E. J. ADVANCED DYNAMICS OF A SYSTEM OF RIGID BODIES (6th edition). Dover 1955. (Macmillan 1905.)

Taylor, J. W. R. A.B.C. OF HELICOPTERS. Ian Allan Ltd. 1955.

Templin, R. L. FATIGUE OF ALUMINIUM. (H. W. Gillett Memorial Lecture 1954.) A.S.T.M. 1954.

U.S. Weather Bureau. FLYING WEATHER FORECASTS: HOW USEFUL ARE THEY? Aviation Series No. 1. U.S.G.P.O. 1955.

U.S. Weather Bureau. ICE ON AIRCRAFT: ITS CAUSES AND EFFECTS. Aviation Series No. 2. U.S.G.P.O. 1955.

U.S. Weather Bureau. THE JET STREAM: A BAND OF VERY FAST WINDS FOUND AT HIGH ALTITUDES. Aviation Series No. 3. U.S.G.P.O. 1955.

U.S. Weather Bureau. TURBULENCE: ITS CAUSES AND EFFECTS. Aviation Series No. 4. U.S.G.P.O. 1955.

I.A.S. Preprints

Papers presented at the Fifth Anglo-American Aeronautical Conference, Los Angeles, 20th-24th June 1955.

No.		
550	The Interaction Between Shock Waves and Boundary Layers	D. W. Holder
551	Power Control Systems for Aircraft	J. W. Ludwig
552	Design of Large Helicopters	B. Kelley
553	Power Plants for Supersonic Flight	E. S. Moult
554	Hypersonic Flow	L. Lees
555	Some Results of the Princeton University Smoke Flow Visualization Program	D. C. Hazen
556	Stress Analysis of Multi-Web Boxes	W. S. Hemp
557	On the Behaviour of Boundary Layers at Supersonic Speeds	R. J. Monaghan
558	Some Influences of Equipment Installations and Systems on Aircraft Design	C. F. Joy
559	Jet Noise	F. B. Greatrex
560	Low Consumption Turbine Engines	A. A. Lombard
561	Combustion for Aircraft Engines	W. T. Olson
562	Fatigue of Aircraft Pressure Cabins	P. B. Walker
563	Design of High Speed Aircraft	E. H. Heinemann
564	Contributions to Fluid Mechanics from Shock Tube Research	I. I. Glass & G. N. Patterson
565	Airline Use of Elementary Statistical Methods in Aircraft Performance Measurement	W. C. Mentzer & F. S. Nowlan
566	Inlet Duct-Engine Flow Compatibility	J. S. Alford
567	Operating Experience with Turbo-Prop Aircraft	P. G. Masefield

Reports

AERODYNAMICS

BOUNDARY LAYER
See INTERNAL FLOW and WINGS AND AEROFOILS

COMPRESSIBLE FLOW
See WINGS AND AEROFOILS

CONTROL SURFACES
See STABILITY AND CONTROL

FLUID DYNAMICS

The fluid flow associated with the impact of liquid drops with solid surfaces. P. Savic and G. T. Boult. National Research Council of Canada. Report No. MT-26. 1955.

A theory is presented of the fluid flow associated with the impact of liquid drops with a solid surface. The shape of the spreading drop is calculated as well as the pressure distribution over the impact plane. Experiments carried out with high speed spark camera bear out the main features predicted by the theory.—(1.4 × 1.9.1).

Contribution a la théorie cinétique des écoulements unidimensionnels. J. J. Bernard. O.N.E.R.A. Publication No. 74 (1955). —(1.4).

Variation of local liquid-water concentration about an ellipsoid of fineness ratio 10 *moving in a droplet field. Rinaldo J. Brun and Robert G. Dorsch. N.A.C.A. Tech. Note* 3410. *April* 1955.

Trajectories of water droplets about an ellipsoid of revolution (10 per cent. thick) in flight through a droplet field were computed with the aid of a differential analyser. Analyses indicate that the local concentration of liquid water at various points about an ellipsoid varies considerably and under some conditions may be several times the free-stream concentration. Curves of the local concentration factor as a function of spatial position are presented in terms of dimensionless parameters that describe flight and atmospheric conditions.—(1.4).

INTERNAL FLOW

Compressor cascade flutter tests. 20° *camber blades, medium and high stagger cascades. D. A. Kilpatrick and J. Ritchie. A.R.C. C.P. No.* 187. (*December* 1953, *published* 1955).

The tests show the existence of three main zones of flutter at high stress; stalling flutter, shock-stalling flutter and choking flutter. These zones are similarly located (with reference to the aerodynamic characteristics) for both the medium and the high stagger cascades tested, and they extend over a wide range of incidence. Good correlation between the zones of flutter and the experimentally measured blade force derivatives, with respect to Mach number and incidence has been obtained. More experimental data are, however, required before a quantitative analysis of the problem can be achieved.—(1.5.4.2).

Theory of the jet syhon. B. Szczeniowski. N.A.C.A. T.N. 3385 (*May* 1955).

A new approach to the theory of the mixing of two currents in an injector is presented dealing with an incompressible ideal fluid. The theory shows new potentialities in an appropriate shaping of the form of the walls of the mixing zone so as to improve the jet-syphon efficiency beyond that heretofore predicted theoretically. Some examples of ways to improve jet-syphon efficiency are indicated.—(1.5.1).

An investigation of several N.A.C.A. 1*-series nose inlets with and without protruding central bodies at high-subsonic Mach numbers and at a Mach number of* 1·2. *R. E. Pendley and H. L. Robinson. N.A.C.A., T.N.* 3436 (*May* 1955).

Measurements of pressure distribution drag, and internal flow pressure loss are presented for three nose inlets, two of which were fitted with protruded central bodies. Test Mach number and inlet-velocity ratio ranged from 0·4 to 1·2 and from 0 to 1·34 respectively. The nose-inlet pressure drag at a Mach number of 1·2 and the central body effects on subcritical drag, the supercritical drag rise, and the inlet total pressure loss are discussed.—(1.5.1.1).

Preliminary investigation of a submerged air scoop utilizing boundary-layer suction to obtain increased pressure recovery. M. R. Nichols and P. K. Pierpont N.A.C.A., T.N. 3437 (*April* 1955).

Results are given of low-speed tests of a submerged inlet consisting essentially of a conventional scoop located in a dimple in the fuselage surface. Boundary layer control systems investigated are shown to provide important increases in performance. It appears that the flow instability frequently encountered in the case of twin internally coupled inlets will be avoided for design high speed inlet-velocity ratios as low as 0·5.—(1.5.1 × 1.1.5).

LOADS

See STABILITY AND CONTROL and TESTING AND INSTRUMENTS

STABILITY AND CONTROL

Low-speed tunnel model tests on tailplane rolling moments in sideslip. A. Spence, J. W. Leathers and D. A. Kirby. R. & M. 2941 (*October* 1951, *published* 1955).

To provide data for stressing purposes, measurements have been made of the effect of sideslip on the rolling moment on a 41·5° swept-back tailplane mounted at three heights on the fin of a model of a single jet aircraft with a 40° swept-back wing. Incidence and tailplane setting were varied, and the effects of rudder deflection were obtained with the tailplane at the top of the fin. Brief results on a delta aircraft model with a delta tailplane at the top of the fin are also included. —(1.8.1 × 1.6.1 × 1.3.3).

Matrix methods for determining the longitudinal-stability derivatives of an airplane from transient flight data. J. J. Donegan. N.A.C.A. Report 1169. (1954).

Three methods, illustrated by several examples using flight data, are presented for calculating the longitudinal stability derivatives from transient flight data. The results indicate the scatter which is typical of this type of analysis.—(1.8.2.1).

WINGS AND AEROFOILS

Some visual observations of the flow over a swept-back wing in a water tunnel, with particular reference to high incidences. N. C. Lambourne and P. S. Pusey. A.R.C., C.P. No. 192 (*May* 1954, *published* 1955).

A qualitative investigation has been made consisting of visual observations of the flow over a small sweptback wing. The tests were restricted to low Reynolds numbers (about 10^5 based on mean chord). Visualisation was effected by (*i*) the addition of small aluminium particles to the water, and (*ii*) the introduction of air into the stream close to the model. In addition, the surface patterns produced by the flow after the wing had been coated with oil were obtained. Attention has been mainly concentrated on the type of flow that exists at high incidence.—(1.10.1.2).

The zero-lift wave drag of a particular family of unswept, tapered wings with linearly varying thickness ratio. A. Henderson and J. M. Goodwin. N.A.C.A., T.N. 3418 (*May* 1955).

On the basis of linear theory, the zero-lift wave drag of a particular family of unswept, tapered wings with linearly varying thickness ratio and symmetrical parabolic-arc sections has been calculated. By comparing the drag for these wings with that for a corresponding constant-thickness ratio wing with rhombic sections, it is found that the variable-thickness ratio wings can be used to advantage with no serious structural penalties if the wings are assumed to have the same given root thickness ratio or the same internal volume.—(1.10.1.3 × 1.2.3).

Exploratory investigation of an airfoil with area suction applied to a porous, round trailing edge fitted with a lift-control vane. R. E. Dannenberg and J. A. Weiber. N.A.C.A., T.N. 3498 (*April* 1955).

A two-dimensional wind tunnel investigation has been made of an N.A.C.A. 65_1-012 aerofoil modified with a porous, round trailing edge fitted with a small vane. Area suction

NOTE.—*The figures in parenthesis at the end of each Summary are for office use only.*

was applied and the vane was used to fix the rear stagnation point. The effects of variations of the chordwise extent of suction and of the vane geometry were investigated with the aim of attaining high maximum lift with low suction quantity.—(1.10.2.1 × 1.1.5).

Optimum two-dimensional multiplanes in supersonic flow. R. M. Licher. Douglas Report No. SM-18688 (January 1955).

Biplanes and other multiplanes are studied by linear theory to find optimum configurations having minimum drag due to lift in supersonic flight. It is shown that suitable camber can often reduce the drag of a two-dimensional biplane. It is sometimes possible to further reduce the drag below that of the cambered biplanes by introduction of a thickness distribution. A general criterion is developed to indicate when the optimum distribution of camber and thickness for minimum drag due to lift has been obtained. Distributions of camber and thickness having no net lift or drag, called zero loadings, are discussed. It is shown how zero loadings may be combined with optimum distributions of camber and thickness to provide more depth for structure without increasing the drag.—(1.10.1.1 × 1.2.3).

Notes on the variation of drag with Mach number of a Busemann biplane. B. J. Beane. Douglas Report No. SM-18737 (January 1955).

It has been observed that the Busemann biplane, at zero lift, has zero wave drag at infinitely many Mach numbers. This result prompted further study of the wave drag of this configuration, with the object of finding the wave drag variation with Mach number.—(1.10.1.1 × 1.2.3).

A study of interference and lift transfer for a staggered delta-wing biplane in supersonic flow. B. J. Beane. Douglas Report No. SM-18738 (January 1955).

The problem of interference and lift transfer in supersonic flow is studied for a biplane having identical delta plan form wings. The wings are staggered so that there is interference at the upper wing only, while the lower wing is forward of the region of influence of the upper wing. Curves are included which show the downwash induced by the lower wing along the centre line of the upper wing.—(1.10.1.1 × 1.2.3).

TESTING AND INSTRUMENTS

The calculation of the wave drag of an arbitrary slender body by means of an electrical analogy tank. P. J. Pocock. N.A.E. Laboratory Report LR-127 (March 1955).

In view of the analogy between the supersonic wave drag of a slender body and the subsonic induced drag of an "equivalent" lifting line the electric analogy tank for solving the Prandtl lifting line equations can be used to calculate the wave drag of arbitrary slender bodies. A Malavard type electric analogy tank at the N.A.E. is described and its use illustrated by examples.—(1.12.6 × 1.6.1).

The N.R.C. icing wind tunnels and some of their problems. C. K. Rush. N.A.E. Laboratory Report LR-133 (April 1955).

The 4½ ft. and blower icing tunnels of the N.R.C. are described. Some of the difficulties encountered due to the icing function and the methods of dealing with them are discussed. The nature of tests undertaken in the tunnels is outlined and a few typical tests described.—(1.12.1.1).

AIRCRAFT OPERATION

Theoretical analysis of light plane landing and take-off accidents due to encountering the wakes of large airplanes. Z. O. Bleviss. Douglas Report No. SM-18647 (December 1954).

It is shown in this analysis that many accidents to light aircraft during landing and take-off are caused by the rotational field of air associated with the trailing vortices from the wings of nearby large aircraft. Detailed numerical results are given for the case of a 40 ft. span light aeroplane with a DC-6B as the large aircraft.—(5.3).

AIRPORTS

A T.V.O.R. automatic-rebroadcast system. T. S. Wonnell and B. H. Boyle. C.A.A Technical Development Report No. 232 (May 1955).

This report describes the development and testing of a T.V.O.R. automatic rebroadcast system developed at the Technical Development and Evaluation Centre, Indianapolis, Indiana. The system provides a means whereby pilots can communicate with one another over airports which have no control tower functioning.—(6.5).

Analytical and simulation studies of terminal-area air traffic control. S. M. Berkowitz and E. L. Fritz. C.A.A. Technical Development Report No. 251 (May 1955).

This report summarises the programme of terminal-area simulation work completed jointly at the Franklin Institute Laboratories for Research and Development and at the Technical Development and Evaluation Centre of the Civil Aeronautics Administration during a nine-month period. The specific tasks involved the set-up, description, and analysis of the mathematical and physical system analogs and a determination of the inter-action of the many variables involved. Advanced radar-vectoring techniques previously developed to a high degree by T.D.E.C. were employed. As a matter of convenience the Washington National Airport terminal area, which is 40 miles in diameter, was used in establishing certain generalised problems with many specific applications and solutions. Representative high-density traffic samples of either arrivals only or of arrivals and departures, with peak durations of one to four hours, were employed. The experiments included theoretical acceptance rates ranging from 30 to 41 landings per hour and used random arrival rates of 35 aircraft per hour and peak departure rates of 31 aircraft per hour. Three methods of analysing the problems were used; an F.I.L. ideal technique, an F.I.L. graphical space-time technique, and the T.D.E.C. dynamic simulator. Correlation among the three was found to be excellent for the wide variety of traffic samples and of conditions tested.—(6.5 × 26).

INSTRUMENTS AND EQUIPMENT

Moving coil vibration generators and pick-ups. H. M. Nelson. A.R.L./1.54 (June 1954).

The equations for an ideal vibration generator or pick-up are derived and the modifications required to allow for a force/current phase lag or force/displacement phase lag in the spring are indicated. A method is indicated whereby the vibrator constants under working conditions can be obtained by measuring the electrical impedance of the vibrator when locked, free and with added mass.—(18.1).

A self-excited, alternating-current, constant-temperature hot-wire anemometer. C. E. Shepard. N.A.C.A., T.N. 3406 (April 1955).

The hot-wire anemometer described herein was used to study turbo-jet-engine compressor rotating stall and surge. The system was capable of measuring flow over a frequency range of zero to 500 cycles per second. A self-sustaining, 8 kilocycle-per-second oscillation was used to heat the wire in a constant temperature system. The relatively rugged, large-diameter wire allowed the use of the anemometer during the full-scale performance testing of compressors and turbo-jet engines. The commercial audio-frequency amplifier used reduced the cost and the time required to build the anemometer.—(18.1).

MATERIALS

A method for assessing the corrosion of iron in soils. Part 1. Theory of the soil probe. K. F. Lorking. A.R.L./Met.12 (December 1954).

A detailed explanation is given of corrosion processes occurring on iron in aqueous media. A description is given of the theoretical basis of methods for the interpretation of potentials assumed by iron and by platinum electrodes. An account is then given of a soil probe designed to assess rates of corrosion of iron in soils.—(21.2.1 × 21.6.1 × 21.6.2).

FATIGUE

See also STRUCTURES

Scatter in fatigue life of 24S-T alclad specimens with drilled holes. W. Weibull. S.A.A.B. T.N. 32 (May 1955).

The fatigue lifetimes of 1,088 flat 24S-T Alclad specimens with two drilled holes have been determined at a tension pulsating between 14 kg. sq. mm. and zero.—(31.2.2.3.2.1.8).